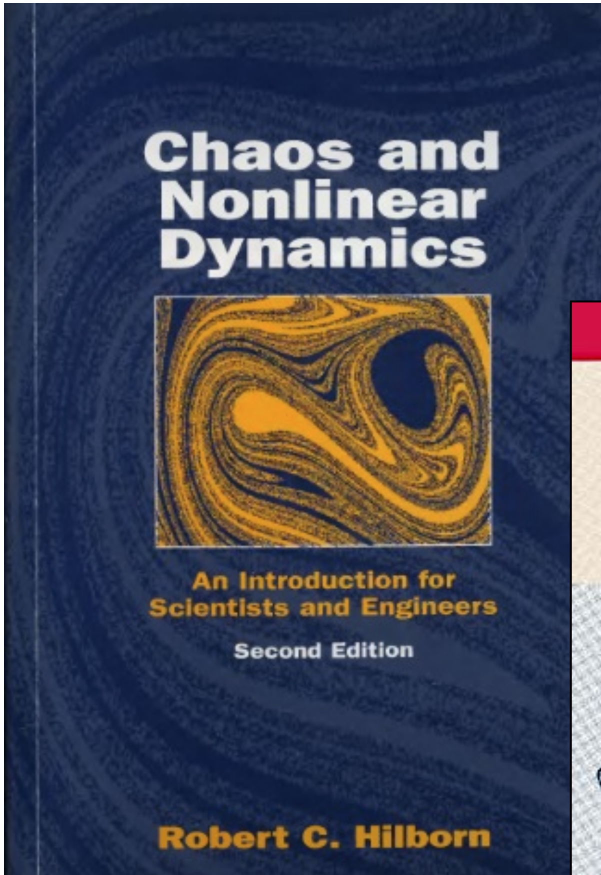
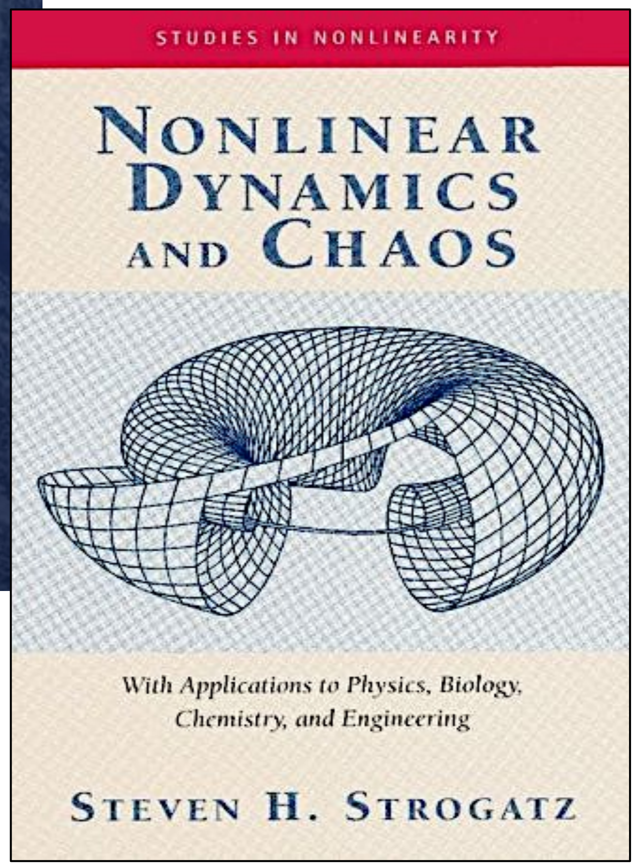
Seconda Parte

Sistemi Dinamici a pochi gradi di libertà





Classificazione dei Sistemi Dinamici

Sistemi dinamici continui (Flussi)

$$\dot{X} = f(X)$$

- **Flussi Dissipativi**
  - Attrattori
    - 1D: Punto fisso
    - 2D: Ciclo Limite
    - 3D: Caotici
- **Flussi Hamiltoniani**
  - Orbite
    - Periodiche
    - Quasi Periodiche
    - Caotiche

Sistemi dinamici discreti (Mappe)

$$x_{n+1} = Ax_n(1 - x_n) \equiv f_A(x)$$

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Sistemi Dinamici Continui
Flussi

3.1 Introduction

In this chapter we will begin to build up the theoretical framework needed to describe more formally the kinds of complex behavior that we learned about in Chapters 1 and 2. We will develop the formalism slowly and in simple steps to see the essential features. We will try to avoid unnecessary mathematical jargon as much as possible until we have built a firm conceptual understanding of the framework.

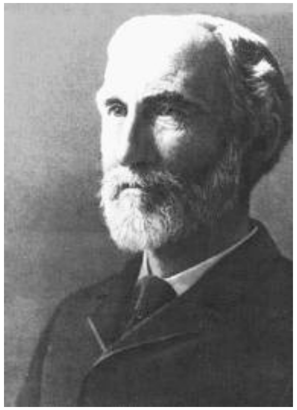
The key theoretical tool in this description is a ***state space*** or ***phase space*** description of the behavior of the system. This type of description goes back to the French mathematician Henri Poincaré in the 1800s and has been widely used in statistical mechanics since the time of the American physicist J. Willard Gibbs (about 1900) even for systems that are linear and not chaotic [Gibbs, 1902]. Of course, we are most interested in the application of state space ideas to nonlinear systems; the behavior of linear systems emerges as a special case.

Sistemi Dinamici → **Meccanica Statistica**

Pochi gradi di libertà

Moltissimi gradi di libertà

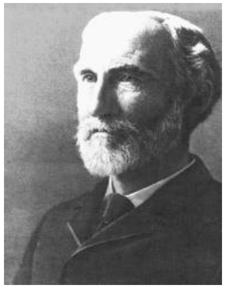
J.H.Poncaré (1854-1912)



J.W.Gibbs (1839-1903)

Spazio degli Stati o Spazio delle Fasi?

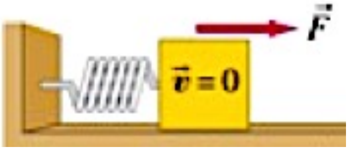
J.H.Poincaré



J.W.Gibbs

Two notes on terminology:

1. In the literature on dynamical systems and chaos, the terms *phase space* and *state space* are often used interchangeably. The term phase space was borrowed from Josiah Willard Gibbs in his treatment of statistical mechanics. The use of this notion in dynamical systems and chaos is somewhat more general than that used by Poincaré and Gibbs; so, we prefer (and will use) the term *state space*.
2. There is also some ambiguity about the use of the term *degree of freedom*. In the classical mechanics of point particles a degree of freedom refers to a pair of variables, such as the position coordinate along the x axis and the corresponding component of the linear momentum p_x. In this usage, a simple mass on a spring has one degree of freedom. (We shall use this definition in Chapter 8.) In dynamical systems theory, the number of degrees of freedom is usually defined as the number of independent variables needed to specify the dynamical state of the system (or alternately, but equivalently, as the number of independent initial conditions that can be specified for the system). We will use the latter definition of degree of freedom (except in Chapter 8). In the first sense of "degrees of freedom," the corresponding phase space must always have an even number (2, 4, 6, …) of dimensions. However, in the theory of dynamical systems and chaos, it will often be useful to have state spaces with an odd number of dimensions. The Lorenz model of Chapter 1 is one such example.



3.2 State Space

In Chapter 1, we introduced rather casually the notion of a state space description of the behavior of a dynamical system. Now we want to develop this notion more carefully and in more detail. Let us start with a very simple example: the motion of a point mass on an ideal (Hooke's Law) spring, oscillating along the x axis. For this system, Newton's Second Law ($\vec{F} = m\vec{a}$) tells us that

R.Hooke (1635-1703)

Oscillatore Armonico

$$F_x = m\frac{d^2x}{dt^2} = -kx \tag{3.2-1}$$

where (as in Chapter 1) k is the spring constant, and m is the particle's mass. The motion of this system is determined for all time by specifying its position coordinate x at some time and its velocity

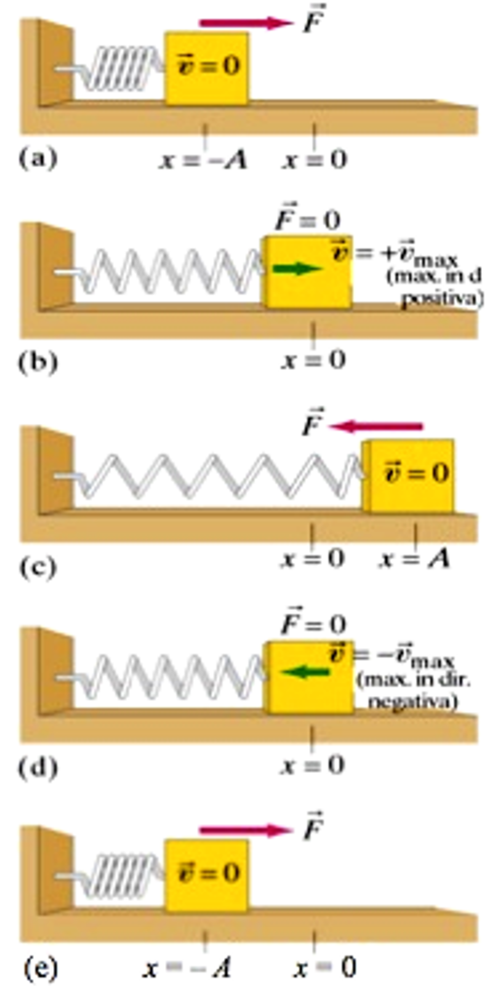
$$\dot{x} = \frac{dx}{dt} \tag{3.2-2}$$

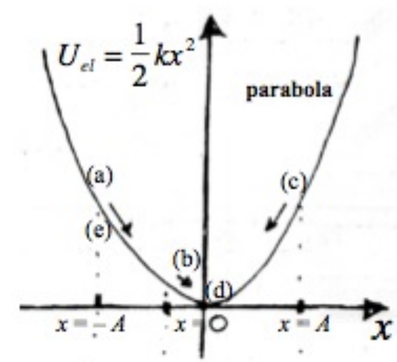
at some time. Traditionally, we choose $t = 0$ for that time, and $x(t = 0) \equiv x_0$ and $dx/dt(t = 0) \equiv \dot{x}_0$ are the "initial conditions" for the system. The motion, in fact, is given by the equation

$$x(t) = x_0 \cos\omega t + \frac{\dot{x}_0}{\omega}\sin\omega t \tag{3.2-3}$$

where $\omega = \sqrt{k/m}$ is the (angular) frequency of the oscillations. By differentiating Eq. (3.2-3) with respect to time, we find the equation for the velocity

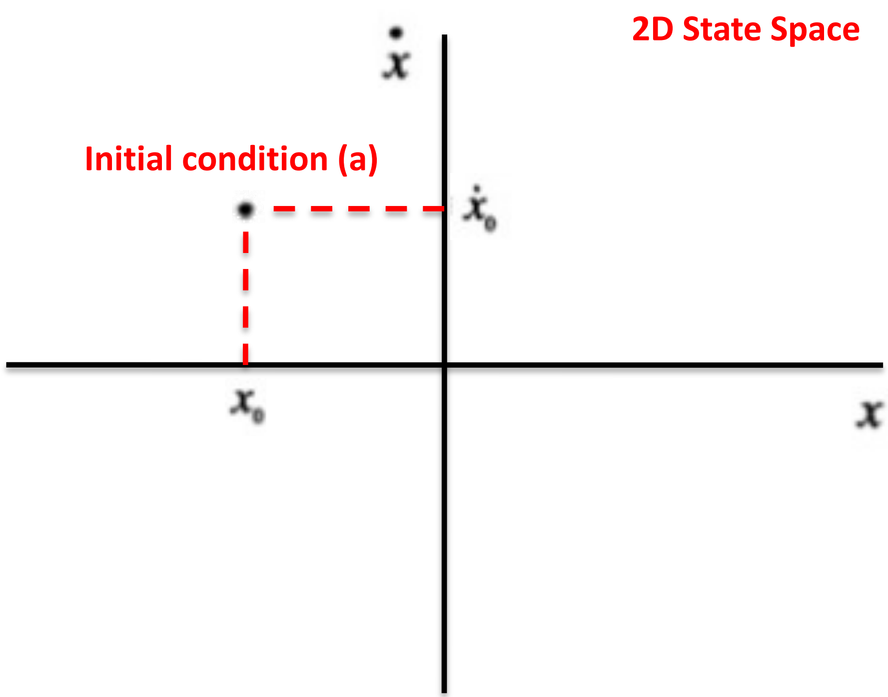
$$\dot{x}(t) = -\omega x_0 \sin\omega t + \dot{x}_0 \cos\omega t \tag{3.2-4}$$

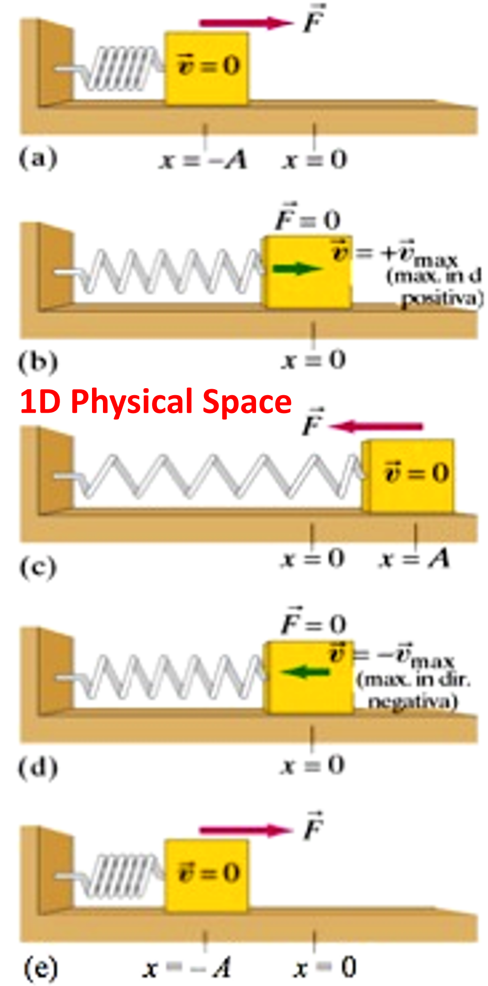


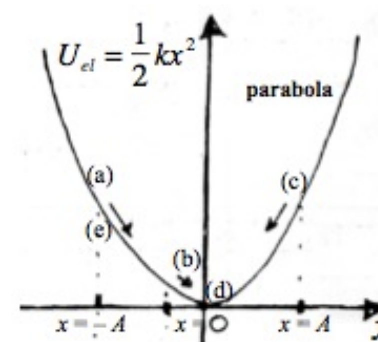


Since knowledge of $x(t)$ and $\dot{x}(t)$ completely specifies the behavior of this system, we say that the system has "two degrees of freedom." (See the comment on terminology on the next page.) At any instant of time we can completely specify the state of the system by indicating a point in an $\dot{x}$ versus x plot. This plot is then what we call the *state space* for this system. In this case the state space is two-dimensional as shown in Fig. 3.1.

Note that the dimensionality of the state space is generally not the same as the spatial dimensionality of the system. The state space dimensionality is determined by the number of variables needed to specify the dynamical state of the system. Our oscillator moves (by construction) in just one spatial dimension, but the state space is two-dimensional.

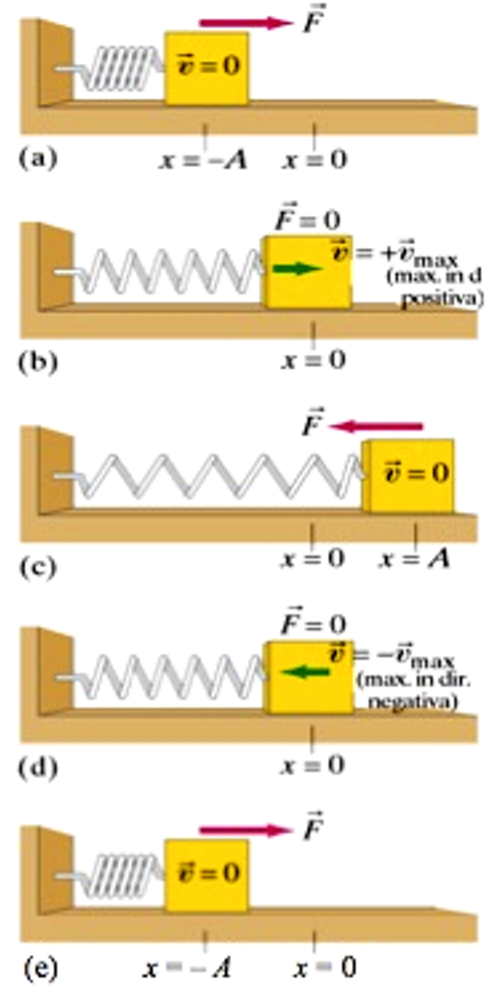






As time evolves, the initial state point in state space follows a trajectory, which, in the case of the mass on a spring, is just an ellipse. (The ellipse can be transformed into a circle by plotting $\dot{x}_0/\omega$ on the ordinate of the state space plot, but that is simply a geometric refinement.) The trajectory closes on itself because the motion is periodic. Such a closed periodic trajectory is called a ***cycle***. Another initial point (not on that ellipse) will be part of a different trajectory. A collection of several such trajectories originating from different initial points constitutes a ***phase portrait*** for the system. Figure 3.1 shows a phase portrait for the mass on a spring system.

A state space and a rule for following the evolution of trajectories starting at various initial conditions constitute what is called a ***dynamical system***. The mathematical theory of such systems is called ***dynamical systems theory***. This theory has a long and venerable history quite independent of the more recent theory of chaos and was particularly well developed by Russian mathematicians (see, for example, [Arnold, 1983]). Because of the extensive groundwork done by mathematicians studying dynamical systems, scientists and mathematicians investigating chaos have been able to make relatively rapid progress in recent years.



Phase Portrait

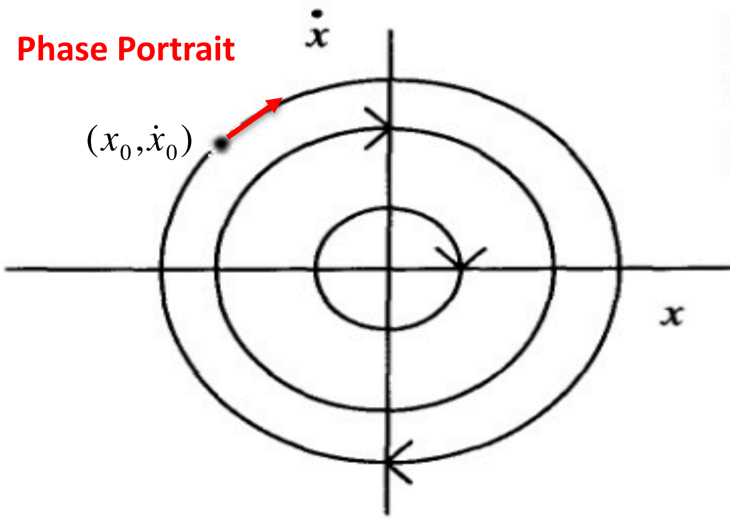
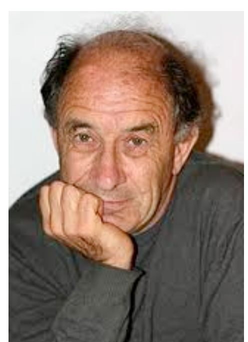
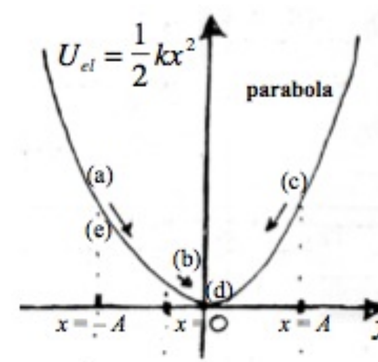


Fig. 3.1. Phase portrait for the mass on a spring. The ellipses are state space trajectories for the system. The larger the ellipse, the larger the total mechanical energy associated with the trajectory.



Vladimir Arnold
(1937-2010)

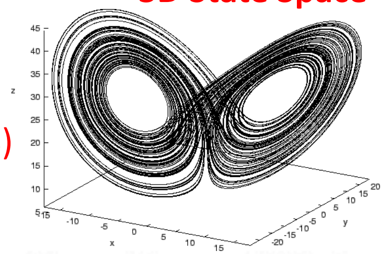


3.3 Systems Described by First-Order Differential Equations

Our theoretical treatment will at first be limited to a special (but rather broad) class of systems for which the equations giving the time-dependence of the state space variables can be expressed as a set of coupled first-order differential equations. To be specific, let us consider a system that has three degrees of freedom (in the second sense described in Section 3.2). Hence, we need three state variables, say, u, v, and w, to describe the state of the system. We will assume that the dynamics of the system can be expressed as a set of three first-order differential equations. That is, the equations involve only the first derivatives of u, v, and w with respect to time:

Ed Lorenz
(1917-2008)

3D State Space



$$\left\{\begin{aligned} \dot{u} &= f(u,v,w) \\ \dot{v} &= g(u,v,w) \\ \dot{w} &= h(u,v,w) \end{aligned}\right. \qquad (3.3\text{-}1)$$

The functions f, g, and h depend on the variables u, v, and w (but not their time derivatives) and also on one or more control parameters, not denoted explicitly. In general u, v, and w occur in all three of f, g, and h, and we say we have a set of "coupled differential equations." Time itself does not appear in the functions f, g, and h. In such a case the system is said to be ***autonomous***. The Lorenz model equations of Chapter 1 are of this form. The time behavior of the system can be tracked by following the motion of a point whose coordinates are $u(t)$, $v(t)$, $w(t)$ in a three-dimensional uvw state space.

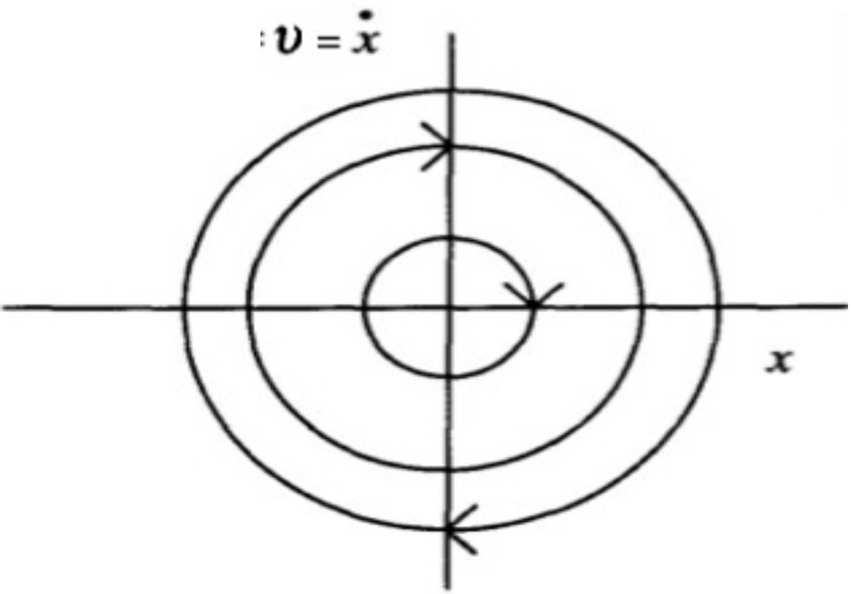
R.Hooke

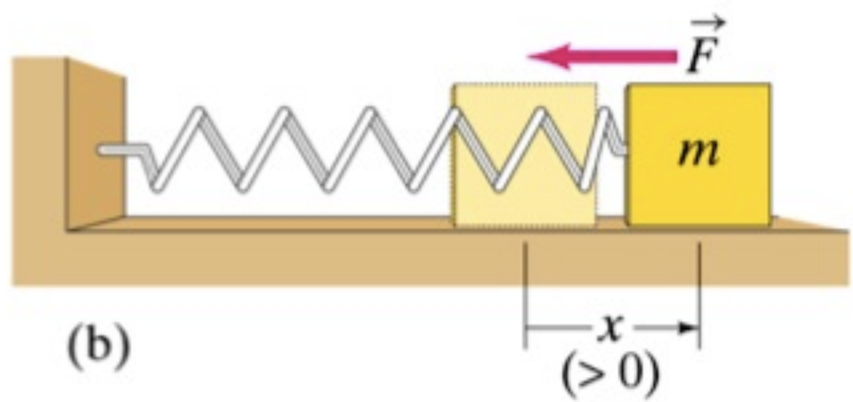
You might note, however, that the mass-on-a-spring model discussed earlier was not of this form. In particular, Eq. (3.2-1) has a second-order time derivative, rather than just a first-order time derivative. However, we can transform Eq. (3.2-1) into the standard form by introducing a new variable, say, υ such that

$$m\frac{d^2x}{dt^2} = -kx \longrightarrow \dot{\upsilon} = \frac{d^2x}{dt^2} \quad \text{accelerazione} \qquad (3.3\text{-}2)$$

Using Eq. (3.3-2) and Eq. (3.2-1), we can write the time evolution equations for the spring system as

$$\left\{\begin{array}{l} \dot{\upsilon} = -\dfrac{k}{m}x \qquad (3.3\text{-}3) \\ \\ \dot{x} = \upsilon \qquad (3.3\text{-}4) \end{array}\right. \rightarrow \ddot{x} = -\frac{k}{m}x$$





We can broaden considerably the class of systems to which Eq. (3.3-1) applies by the following "trick." Suppose that after applying the usual reduction procedure, the functions on the right-hand side of Eq. (3.3-1) still involve the time variable. (In that case, we say the system is ***nonautonomous***.) This case most often arises when the system is subject to an externally applied time-dependent "force." For a two-degree-of-freedom system, the standard equations will be of the form

$$\dot{u} = f(u,v,t) \tag{3.3-5}$$

$$\dot{v} = g(u,v,t) \tag{3.3-6}$$

We can change these equations to a set of autonomous equations by introducing a new variable w whose time derivative is given by

$$\dot{w} \equiv \frac{dt}{dt} = 1 \tag{3.3-7}$$

The dynamical equations for the system then become

$$\begin{aligned} \dot{u} &= f(u,v,w) \\ \dot{v} &= g(u,v,w) \\ \dot{w} &= 1 \end{aligned} \tag{3.3-8}$$

We have essentially enlarged the number of dimensions of the state space by 1 to include time as one of the state space variables. The advantage of this trick is that it allows us to treat nonautonomous systems (those with an imposed time dependence) on the same footing as autonomous systems. The price we pay is the difficulty of treating one more dimension in state space.

Punti Fissi

Why do we use this standard form (first-order differential equations) for the dynamical equations? The basic reason is that this form allows a ready identification of the fixed points of the system, and (as mentioned earlier) the fixed points play a crucial role in the dynamics of these systems. Recall that the *fixed points* are defined as the points in state space for which all of the time derivatives of the state variables are 0. Thus, with our standard form equations the fixed points are determined by requiring that

$$\left\{\begin{aligned} \dot{u} &= f(u,v,w) \\ \dot{v} &= g(u,v,w) \\ \dot{w} &= h(u,v,w) \end{aligned}\right. \longrightarrow \left\{\begin{aligned} f(u,v,w) &= 0 \\ g(u,v,w) &= 0 \\ h(u,v,w) &= 0 \end{aligned}\right. \tag{3.3-9}$$

Thus, we find the fixed points by solving the three (for our three-dimensional example) coupled *algebraic* equations.

An important question: Can the dynamical equations for all systems be reduced to the form of Eq. (3.3-1)? The answer is yes if (and this is an important *if*) we are willing to deal with an infinite number of degrees of freedom. For example, systems that are described by partial differential equations (that is, equations with partial derivatives rather than ordinary derivatives) or systems described by integral-differential equations (with both integrals and derivatives occurring in essential ways) or by systems with time-delay equations (where the state of the system at time t is determined not only by what is happening at that time but also by what happened earlier), all can be reduced to a set of first-order ordinary differential equations, but with an infinite number of equations coupled together.

3.4 The No-Intersection Theorem

Before beginning the analysis of the types of trajectories and fixed points that can occur in state space, we state a fundamental and important theorem:

The No-Intersection Theorem: Two distinct state space trajectories cannot intersect (in a finite period of time). Nor can a single trajectory cross itself at a later time.

By *distinct* trajectories, we mean that one of the trajectories does not begin on one of the points of the other trajectory. The parenthetical comment about a *finite* period of time is meant to exclude those cases for which distinct trajectories approach the same point as $t \to \infty$. (In the excluded case, we say the trajectories approach the point *asymptotically*.)

The basic physical content of this theorem is a statement of *determinism*. We have already mentioned that the state of a dynamical system is specified by its location in state space. Furthermore, if the system is described by equations of the form of Eq. (3.3-1), then the time derivatives of the state variables are also determined by the location in state space. Hence, how the system evolves into the future is determined solely by where it is now in state space. Hence, we cannot have two trajectories intersect in state space. If two trajectories *did* cross at some point, then the two trajectories would have the same values of their state variables and the same values of their time derivatives, yet they would evolve in different ways. This is impossible if their time evolution is described by equations like Eq. (3.3-1). As we shall see, the No-Intersection Theorem highly constrains the behavior of trajectories in state space.

$$\begin{cases} \dot{u} = f(u,v,w) \\ \dot{v} = g(u,v,w) \\ \dot{w} = h(u,v,w) \end{cases}$$

The No-Intersection Theorem can also be based mathematically on uniqueness theorems for the solutions of differential equations. For example, if the functions *f*, *g*, and *h* on the right-hand-side of Eq. (3.3-1) are continuous functions of their arguments, then only one solution of the equations can pass through a given point in state space. [The more specific mathematical requirement is that these functions be continuous and at least once differentiable. This is the so-called Lipschitz condition (see, for example, [Hassani, 1991], pp. 570–71).

We shall see two apparent violations of this theorem. The first occurs for those asymptotic "intersections" mentioned earlier. The second occurs when we project the trajectory onto a two–dimensional plane for the sake of illustration. For example, Fig. 1.19 shows a *YZ* plane projection of a trajectory for the Lorenz model. The trajectory seems to cross itself several times. However, this crossing occurs only in the two–dimensional projection. In the full three–dimensional state space the trajectories do not cross.

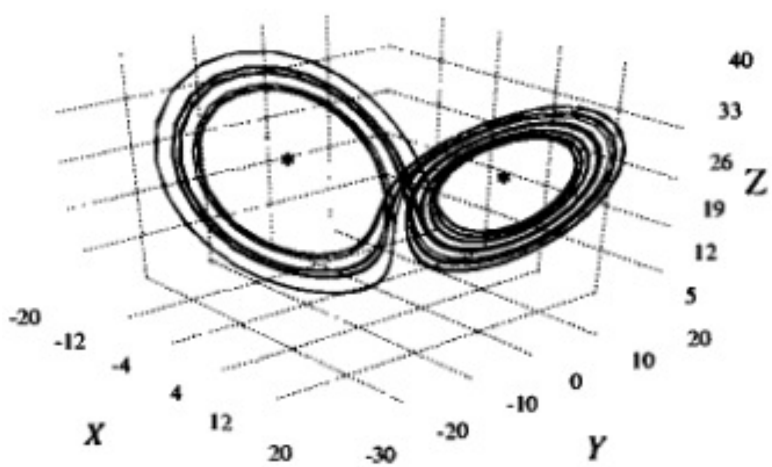
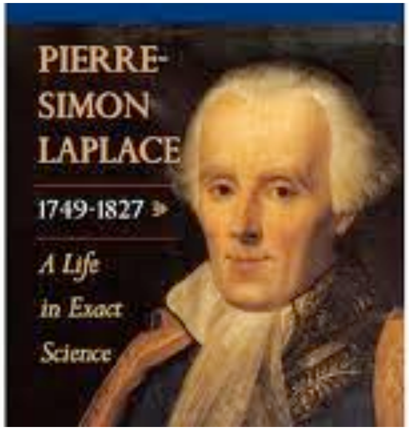


Fig. 1.19.

Il Manifesto di Laplace

Il più noto "manifesto" del determinismo è certamente quello elaborato nel 1812 dal matematico e astronomo francese Pierre S.Laplace: *"Possiamo considerare lo stato attuale dell'universo come l'effetto del suo passato e la causa del suo futuro. Una intelligenza che, per un istante dato, potesse conoscere tutte le forze da cui la natura è animata e la situazione rispettiva degli esseri che la compongono, e che inoltre fosse abbastanza grande da sottomettere questi dati all'analisi, abbraccerebbe nella stessa formula i movimenti del più grandi corpi dell'universo e quelli dell'atomo più leggero: nulla le risulterebbe incerto, l'avvenire come il passato sarebbe presente ai suoi occhi*"

Pierre Simon Laplace, *"Essai philosophique des probabilitàs"* (1812)



Eq. di Hamilton

$$\begin{cases} \dot{P} = -\dfrac{\partial H}{\partial Q} \\ \dot{Q} = \dfrac{\partial H}{\partial P} \end{cases}$$

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condizioni iniziali

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Tempo, Cosmologia e Libero Arbitrio

di Alessandro Pluchino

Saggio semi-divulgativo che si concentra in particolare sulle relazioni tra il problema del Tempo, i nuovi modelli Cosmologici del Multiverso e la nostra sensazione di possedere un Libero Arbitrio, cercando di mostrare come questi tre ambiti, apparentemente distinti tra loro, siano in realtà indissolubilmente legati

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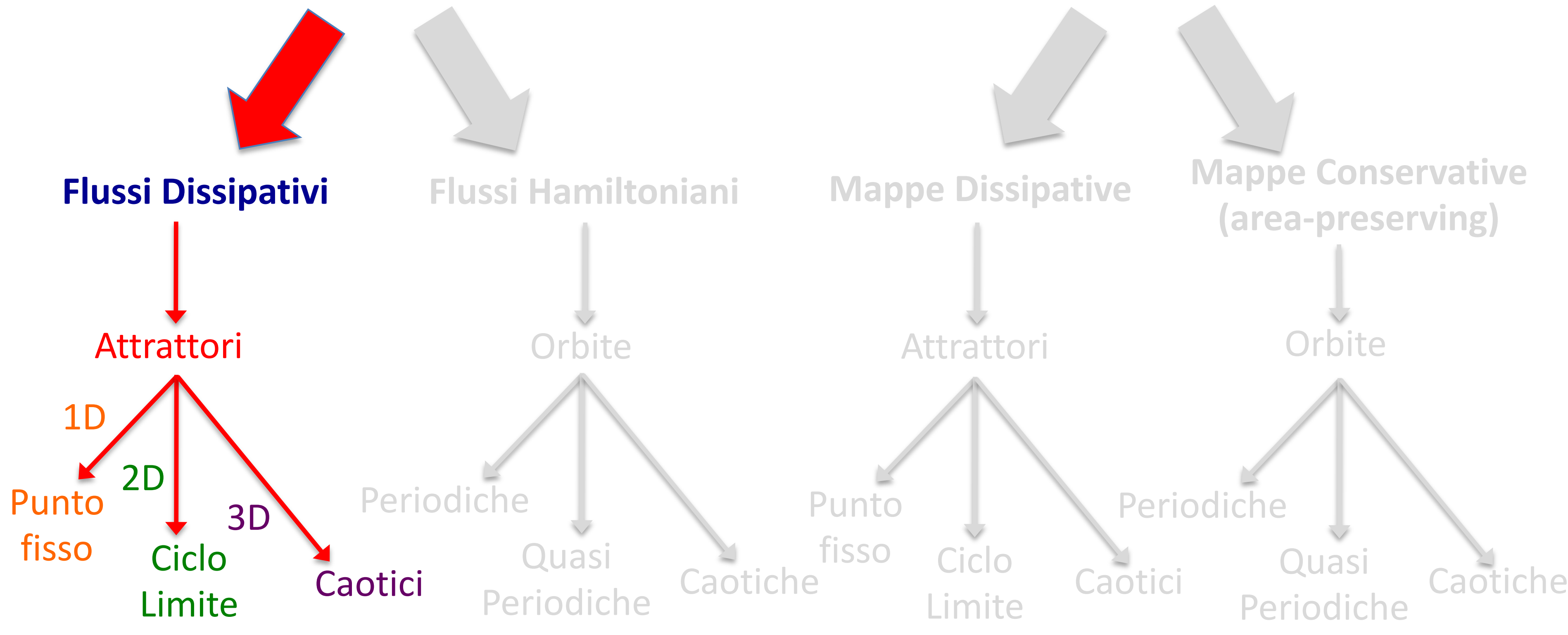
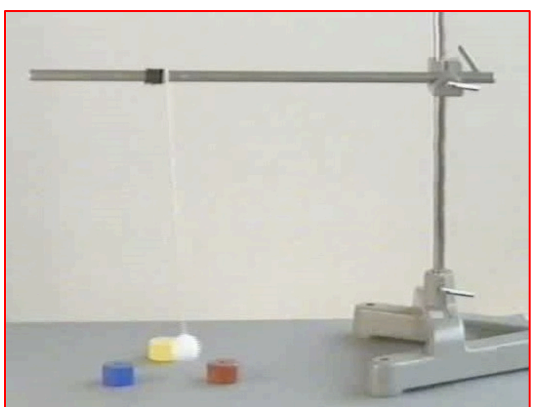


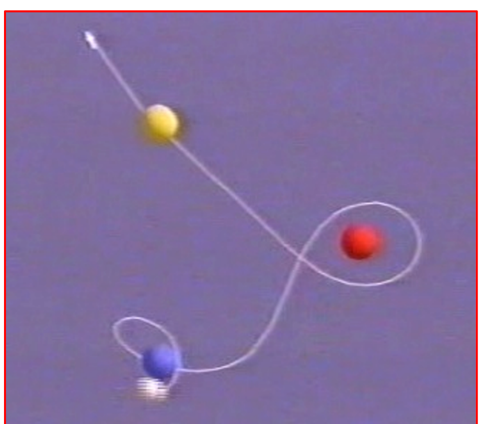
Flussi dissipativi

3.5 Dissipative Systems and Attractors

In our current discussion of state space and its trajectories, we will limit our discussion to the case of dissipative systems. (Systems for which dissipation is unimportant will be discussed in Chapter 8.) As mentioned in Chapter 1, a dissipative system displays the nice feature that the long-term behavior of the system is largely independent of how we "start up" the system. We will elaborate this point in Section 3.9. (Recall, however, that there may be more than one possible "final state" for the system.) Thus, for dissipative systems, we generally ignore the transient behavior associated with the start up of the system and focus our attention on the system's long-term behavior.

As the dissipative system evolves in time, the trajectory in state space will head for some final state space point, curve, area, and so on. We call this final point or curve (or whatever geometric object it is) the ***attractor*** for the system since a number of distinct trajectories will approach (be attracted to) this set of points in state space. For dissipative systems, the properties of these attractors determine the dynamical properties of the system's long-term behavior. However, we will also be interested in how the trajectories approach the attractor.





The set of initial conditions giving rise to trajectories that approach a given attractor is called the ***basin of attraction*** for that attractor. If more than one attractor exists for a system with a given set of parameter values, there will be some initial conditions that lie on the border between the two (or more) basins of attraction. See Fig. 3.2. These special initial conditions form what is called a ***separatrix*** since they separate different basins of attraction.

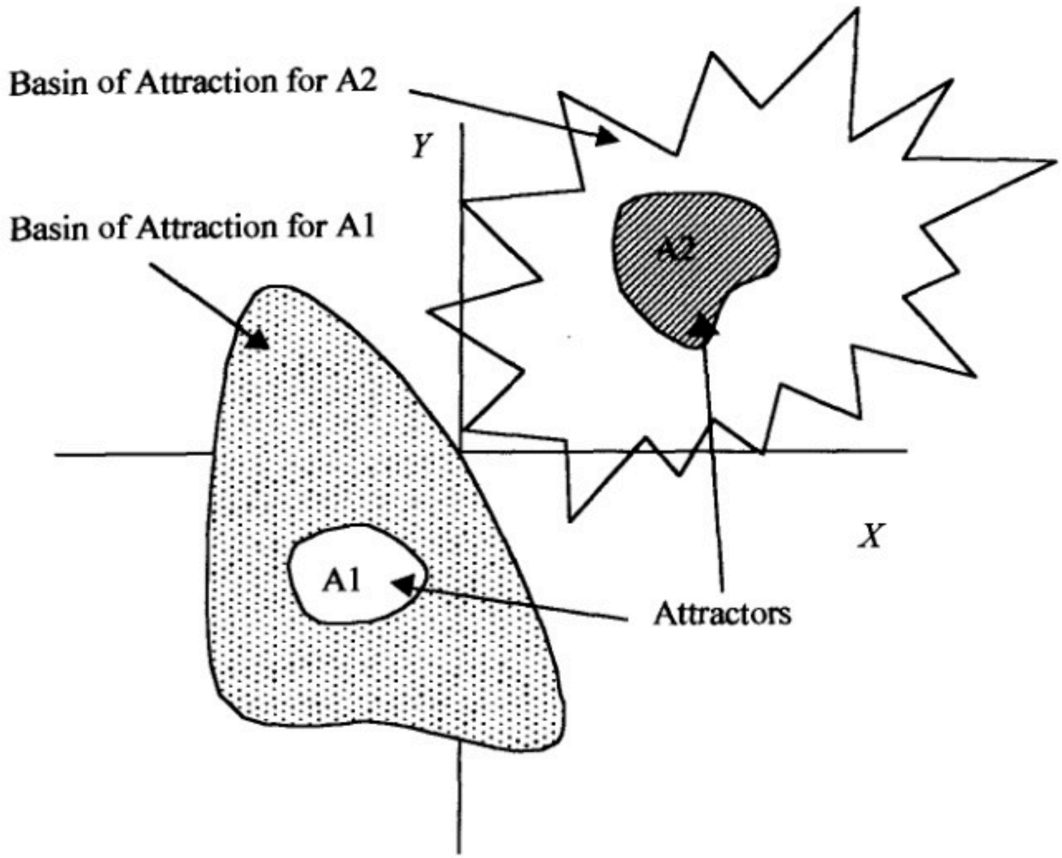
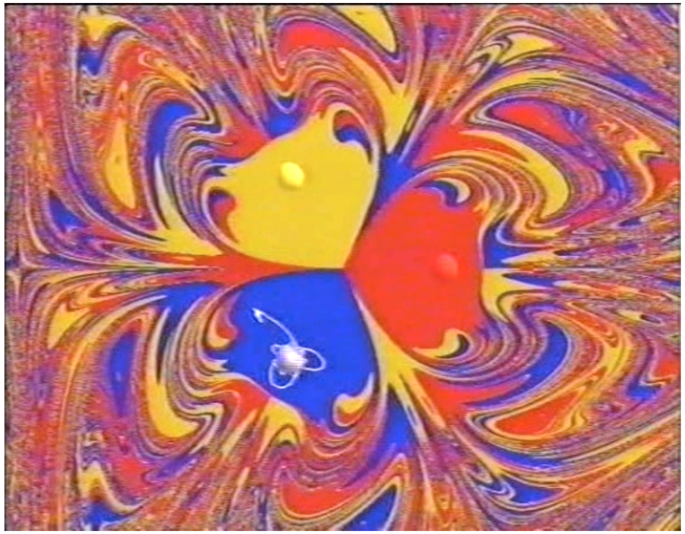
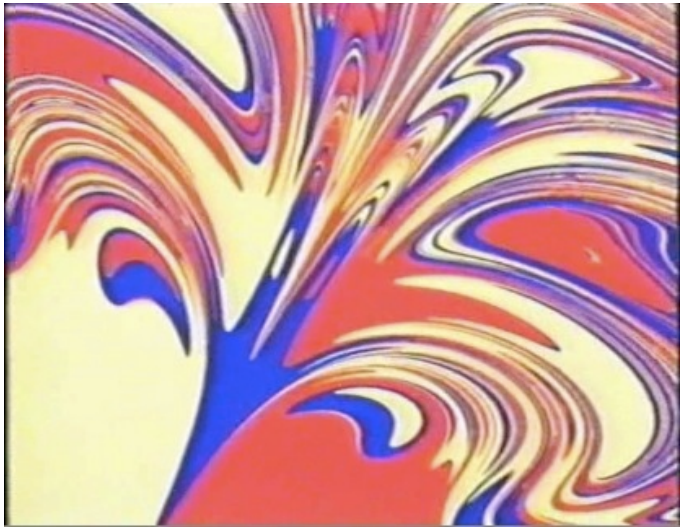


Fig. 3.2. A sketch of attractors A1 and A2 and basins of attraction in state space. Trajectories starting inside the dotted basin eventually end up in the attractor region inside the dotted region. Trajectories starting in the other basin head for the other attractor. For starting points outside these two basins, the trajectories may go toward a third attractor (not shown). The line bounding a basin of attraction forms a separatrix.

The geometric properties of basins of attraction can often be complicated. In some cases the boundaries are highly irregular, forming what are called ***fractal basin boundaries*** (GMO83, MGO85). In other cases, the basins of attraction can crivellato be highly intertwined, forming what are called ***riddled basins of attraction*** (SOO93a): any point in one basin is close to another point in another basin of attraction. As we mentioned in Chapter 1, the existence of such complicated structures means that our ability to predict even which attractor a system will evolve to is severely compromised.





In the next sections we will describe the kinds of trajectories and attractors that can occur in state spaces of different dimensions. The dimensionality of the state space is important because the dimensionality and the No-Intersection Theorem together highly constrain the types of trajectories that can occur. In fact, we shall see that we need at least three state space dimensions in order to have a chaotic trajectory. We will, however, begin the cataloging with one and two dimensions to develop the necessary mathematical and conceptual background.

Classificazione dei Sistemi Dinamici

Sistemi dinamici continui (Flussi)

$$\dot{X} = f(X)$$

- **Flussi Dissipativi**
  - Attrattori
    - 1D: Punto fisso
    - 2D: Ciclo Limite
    - 3D: Caotici
- Flussi Hamiltoniani
  - Orbite
    - Periodiche
    - Quasi Periodiche
    - Caotiche

Sistemi dinamici discreti (Mappe)

$$x_{n+1} = A x_n (1 - x_n) \equiv f_A(x)$$

- Mappe Dissipative
  - Attrattori
    - Punto fisso
    - Ciclo Limite
    - Caotici
- Mappe Conservative (area-preserving)
  - Orbite
    - Periodiche
    - Quasi Periodiche
    - Caotiche

Classificazione dei Sistemi Dinamici

Sistemi dinamici continui (Flussi)

$$\dot{X} = f(X)$$

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    - 1D: Punto fisso
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    - Periodiche
    - Quasi Periodiche
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Sistemi dinamici discreti (Mappe)

$$x_{n+1} = Ax_n(1-x_n) \equiv f_A(x)$$

- **Mappe Dissipative**
  - Attrattori
    - Punto fisso
    - Ciclo Limite
    - Caotici
- **Mappe Conservative (area-preserving)**
  - Orbite
    - Periodiche
    - Quasi Periodiche
    - Caotiche

Flussi dissipativi
in
una dimensione

3.6 One-Dimensional State Space

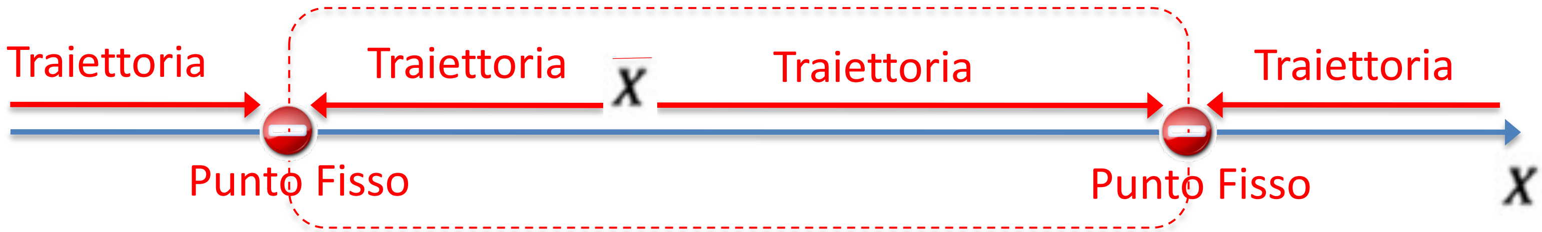
A one-dimensional system, in the sense of dimension we are using here, has only one state variable, which we shall call X. This is, as we shall see, a rather uninteresting system in terms of its dynamical possibilities; however, it will be useful for developing our ideas about trajectories and state space. For this one dimensional state space, the dynamical equation is

$$\boxed{\dot{X} = f(X)} \tag{3.6-1}$$

The state space is just a line: the X axis.

First let us consider the fixed points for such a system, that is the values of X for which $\dot{X} = 0$. Why are the fixed points important? If a trajectory happens to get to a fixed point, then the trajectory stays there. Thus, the fixed points divide the X axis up into a number of "noninteracting" regions. We say the regions are noninteracting because a trajectory that starts from some initial $\bar{X}$ value in a region located between two fixed points can never leave that region.

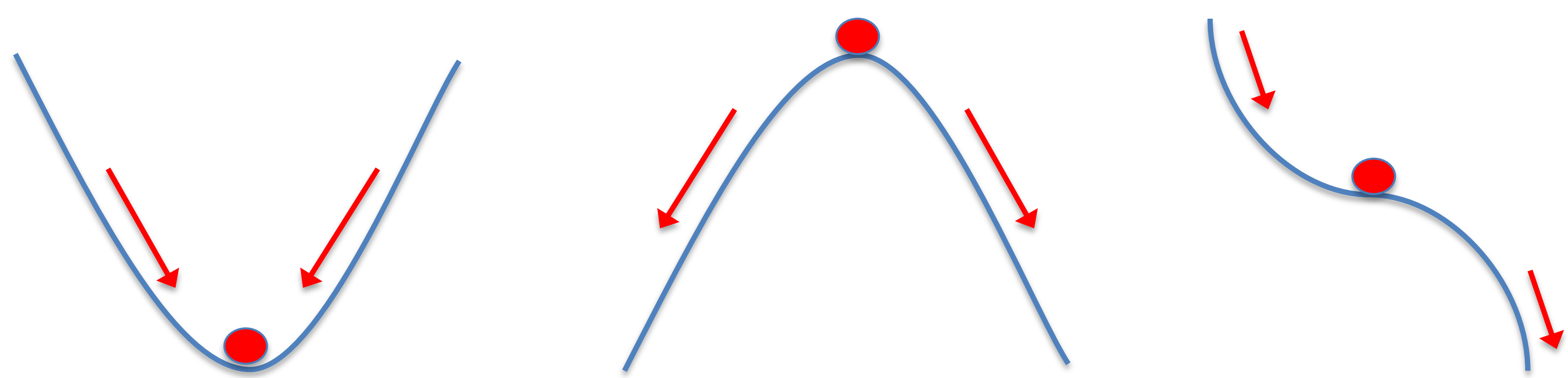
SPAZIO DEGLI STATI 1D



Now we want to investigate what happens to trajectories that are near a fixed point. For a one-dimensional state space, there are three types of fixed points:

1. ***Nodes*** (***sinks***): fixed points that attract nearby trajectories.
2. ***Repellors*** (***sources***): fixed points that repel nearby trajectories.
3. ***Saddle points***: fixed points that attract trajectories on one side but repel them on the other. (The origin of the term *saddle point* will become obvious when we get to the two-dimensional case.)

A node is said to be a ***stable fixed point*** in the sense that trajectories that start near it are drawn toward it much like a ball rolling to a point of stable equilibrium under the action of gravity. A repellor is an example of an ***unstable fixed point*** in analogy with a ball rolling off the top of a hill. The top of the hill is an equilibrium point, but the situation is unstable: The slightest nudge to the side will cause the ball to roll away from the top of the hill. A saddle point attracts trajectories in one direction while repelling them in the other direction.



Node (stable fixed point) **Repellor** (unstable fixed point) **Saddle Point**

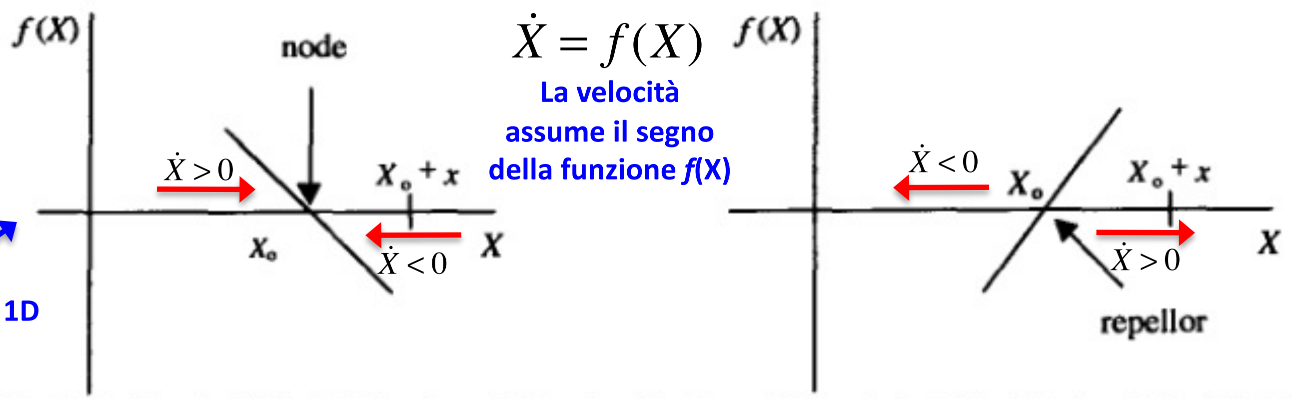
METAFORA GRAVITAZIONALE DEL LANDSCAPE ENERGETICO

Punti Fissi in una dimensione

How do we determine what kind of fixed point we have? The argument goes as follows: Let X_o be the location of the fixed point in question. By definition, we

$$\boxed{\dot{X}\Big|_{X=X_o} = f(X_o) = 0} \tag{3.6-2}$$

Now consider a trajectory that has arrived at a point just to the right of X_o. Let us call that point $X = X_o + x$ (see Fig. 3.3). We shall assume that x is small and positive. If $f(X_o + x)$ is positive (for x positive), then $\dot{X}$ is positive and hence the trajectory point will move away from X_o (toward more positive X values). On the other hand, if $f(X_o + x)$ is negative (for x positive), then $\dot{X}$ is negative and the trajectory moves to the left toward the fixed point X_o. Conversely, if we start to the left of X_o along the X axis, then we need $f(X_o - x)$ positive to move toward X_o and $f(X_o - x)$ negative to move away from X_o. These two cases are illustrated in Fig. 3.3. When trajectories on both sides of X_o move away from X_o, the fixed point is a repellor. When trajectories on both sides of X_o move toward X_o, the fixed point is a node.



LO SPAZIO DEGLI STATI 1D
E' IL SOLO ASSE X

Fig. 3.3. On the left, $f(X)$ in the neighborhood of a node located at X_o. On the right, $f(X)$ in the neighborhood of a repellor located at X_o.

Both of these cases can be summarized by noting that the derivative of $f(X)$ with respect to X evaluated at X_o is negative for a node and positive for a repellor. The value of this derivative at the fixed point is called the ***characteristic value*** or ***eigenvalue*** (from the German *eigen* = characteristic) of that fixed point. We call the characteristic value λ.

$$\boxed{\lambda = \left.\frac{df(X)}{dX}\right|_{X=X_o}} \quad \text{valore caratteristico o autovalore del Punto Fisso} \qquad (3.6\text{-}3)$$

We summarize these results in Table 3.1. The crucial and important lesson here is that we can determine the character of the fixed point and consequently the behavior of the trajectories near that fixed point by evaluating the derivative of the function $f(X)$ at that fixed point.

Table 3.1
Characteristic Values

$\lambda < 0$	fixed point is a node
$\lambda > 0$	fixed point is a repellor

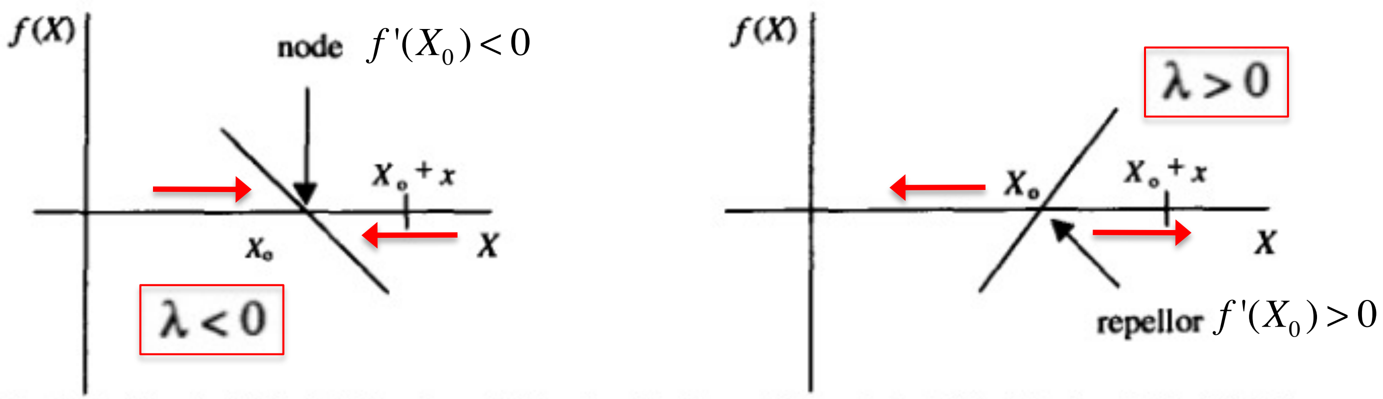
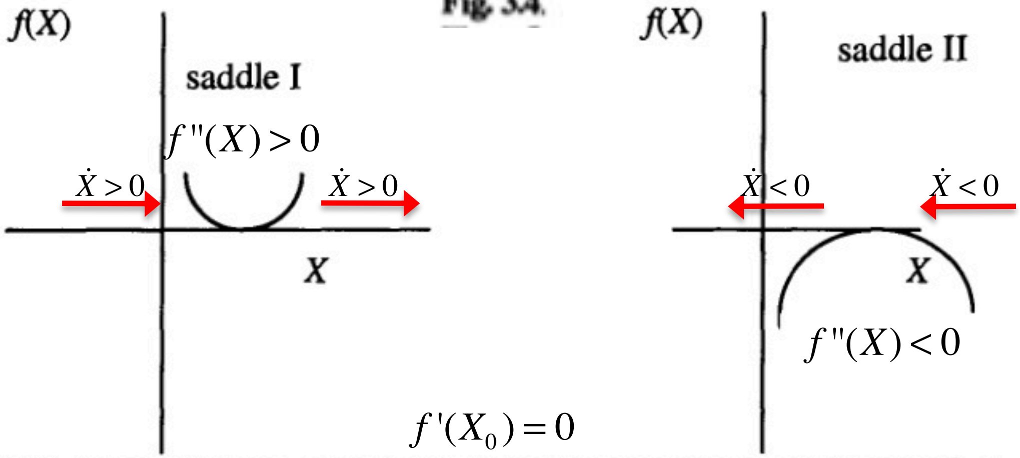


Fig. 3.3. On the left, $f(X)$ in the neighborhood of a node located at X_o. On the right, $f(X)$ in the neighborhood of a repellor located at X_o.

What happens when the characteristic value is equal to 0? The fixed point might be a node or a repellor or a saddle point. To find out which is the case we need to look at the second derivative of f with respect to X as well as the first derivative. For a saddle point, the second derivative has the same sign on both sides of X_o (see Fig. 3.4). Thus, we see that for a saddle point the trajectory is attracted toward the fixed point on one side, but repelled from the saddle point on the other.

Fig. 3.4.



$f'(X_0)=0$

$\boxed{\lambda = 0.}$

For the node and repellor with characteristic value equal to 0, the second derivative changes sign as X passes through X_o (it is positive on the left and negative on the right for the node and negative on the left and positive on the right for the repellor). These kinds of "flat" nodes and repellors attract and repel trajectories more slowly than the nodes and repellors with nonzero characteristic values. For the type I saddle point, trajectories are attracted from the left but repelled on the right. The attraction and repulsion are reversed for the type II saddle points.

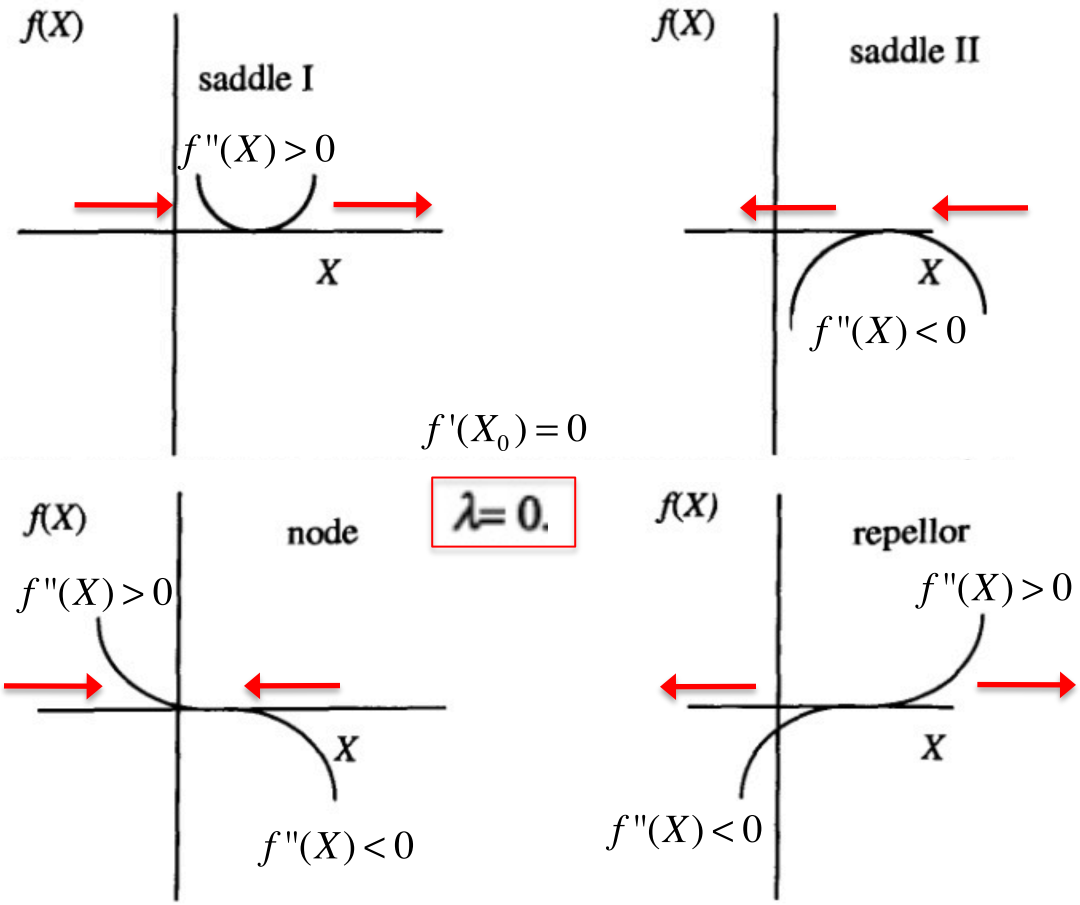


Fig. 3.4. Four possible types of fixed points in one-dimension with characteristic value $\lambda = 0$. These fixed points are structurally unstable.

We will not discuss these types of "flat" fixed points further because, in a sense, they are relatively rare. They are rare because they require both the function $f(X)$ and its first derivative to be 0. If we have only one control parameter to adjust, then it is "unlikely" that we can satisfy both conditions simultaneously for some range of parameter values. In more formal terms, we talk about the ***structural stability*** of the fixed point. If the fixed point keeps the same character when the shape or position of the function changes slightly (for example, as a control parameter is adjusted), then we say that the fixed point is structurally stable. If the fixed point changes character or disappears completely under such changes, then we say it is structurally unstable. For example, the nodes and repellors shown in Fig. 3.3 are structurally stable because shifting the function $f(X)$ up and down slightly or changing its shape slightly does not alter the character of the fixed point.

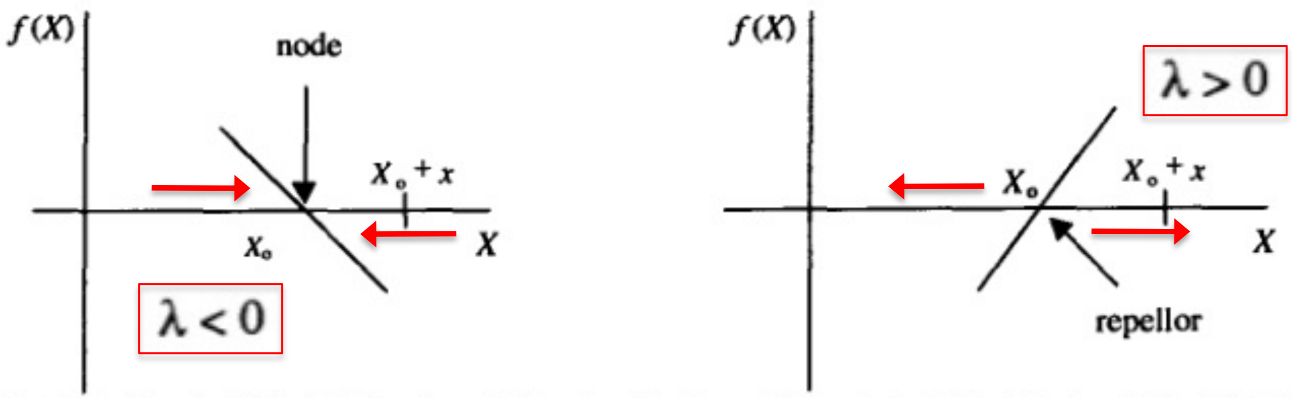


Fig. 3.3. On the left, $f(X)$ in the neighborhood of a node located at X_o. On the right, $f(X)$ in the neighborhood of a repellor located at X_o.

However, the fixed points shown in Fig. 3.4 are structurally unstable. For example, a small change in the function, say, shifting it up or down by a small amount, will cause a saddle point to either disappear completely or change into a node-repellor pair (see Fig. 3.5).

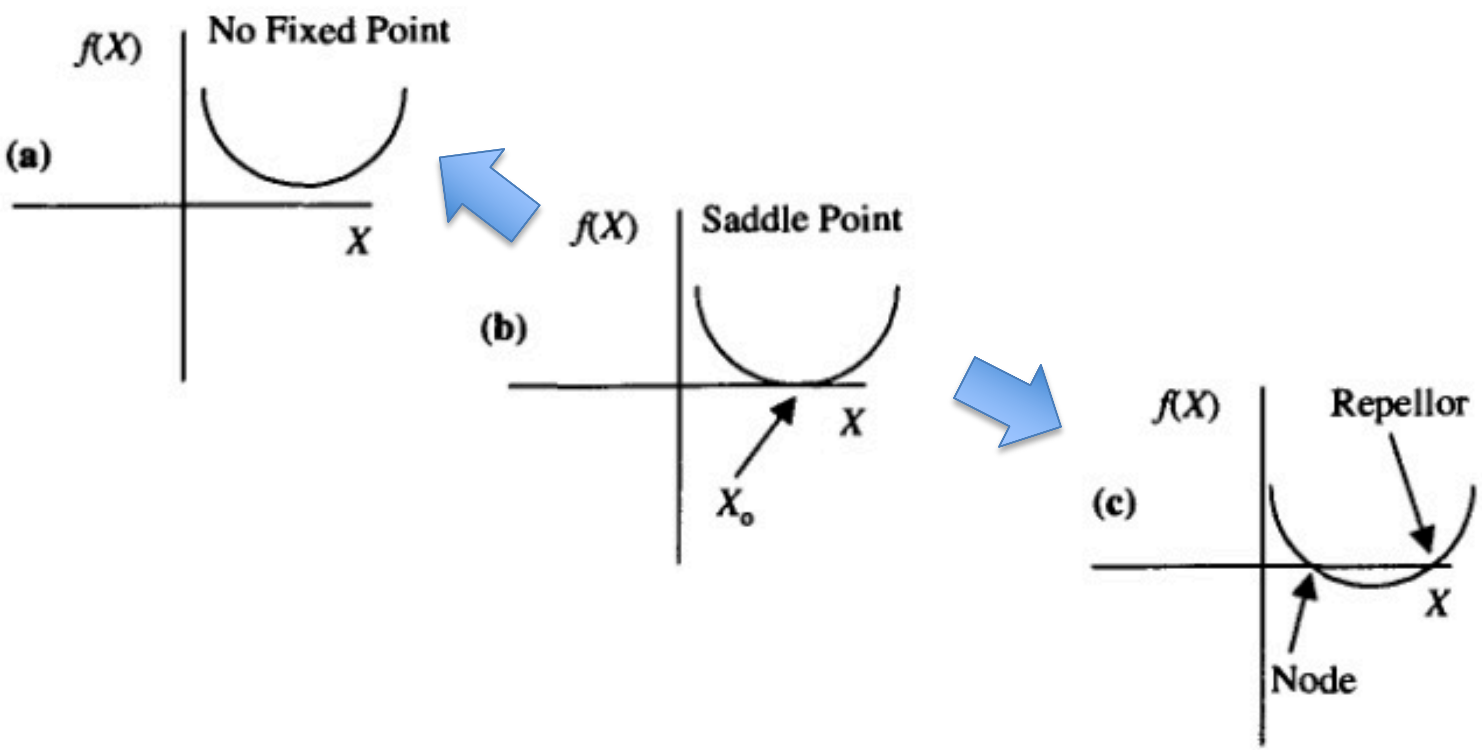
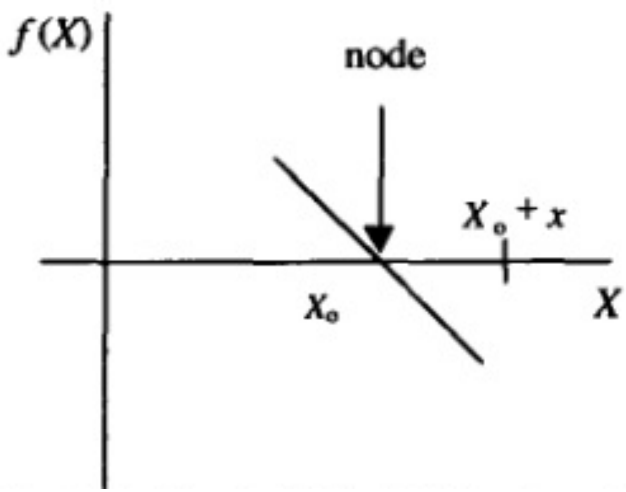


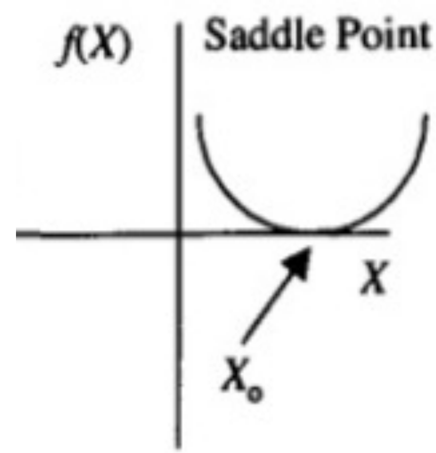
Fig. 3.5. In one-dimensional state spaces, a saddle point, the point X_o in (b), is structurally unstable. A small change in the function $f(X)$, for example pushing it up or down along the vertical axis, either removes the fixed point (a), or changes it into a node and a repellor (c).

To examine in detail what constitutes a small change in the function $f(X)$ and how to decide whether a particular structure is stable or unstable would lead us rather far afield (see [Guckenheimer and Holmes, 1990]). Most of the work in nonlinear dynamics focuses on structurally stable state space portraits because in any real experiment the only properties that we can observe are those that exist for some finite range of parameter values. We can never set the experimental conditions absolutely precisely, and "noise" always smears out parameter values. However, as we shall see, structurally unstable conditions are still important: In many cases they mark the border between two different types of behavior for the system. We will return to this issue at the end of this chapter in the discussion of bifurcations.

Strutturalmente stabile



Strutturalmente instabile



$\boxed{\dot{X} = f(X)}$

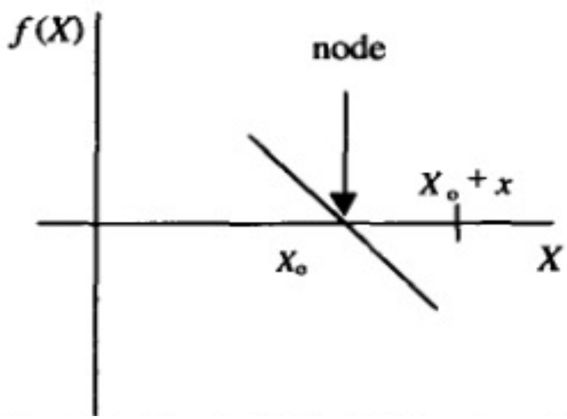
3.7 Taylor Series Linearization Near Fixed Points

The formal discussion of the nature of fixed points can be summarized very compactly using the mathematical notion of a ***Taylor series expansion*** of the function $f(X)$ for X values in the neighborhood of the fixed point X_o:

$$f(X) = \cancel{f(X_o)} + (X - X_o)\frac{df}{dX} \boxed{+ \frac{1}{2}(X - X_o)^2 \frac{d^2 f}{dX^2} + \frac{1}{6}(X - X_o)^3 \frac{d^3 f}{dX^3} + \dots} \quad (3.7\text{-}1)$$

Nelle vicinanze del punto fisso si possono trascurare…

where all the derivatives are evaluated at $X = X_o$. At a fixed point for a dynamical system, the first term on the right-hand side of Eq. (3.7-1) is 0, by the definition of fixed point. The Taylor series expansion tells us that the function $f(X)$ near X_o is determined by the values of the derivatives of f evaluated at X_o and the difference between X and X_o. This information together with the dynamical equation (3.6-1) is sufficient to predict the behavior of the system near the fixed point.



$\boxed{\dot{X} = f(X)}$

In particular, we introduce a new variable $x = X - X_o$, which measures the distance of the trajectory away from the fixed point. If we neglect all derivatives of order higher than the first, then x satisfies the equation

EQUAZIONE LINEARIZZATA

$$f(X) = (X - X_o)\frac{df}{dX} + \ldots \longrightarrow \dot{x} = \left.\frac{df}{dX}\right|_{X_o} x$$

$$\rightarrow \dot{x} = \lambda x \qquad (3.7\text{-}2)$$

$$\rightarrow \frac{dx}{dt} = \lambda x$$

$$\rightarrow \frac{dx}{x} = \lambda dt \qquad (3.7\text{-}3)$$

$$\rightarrow \int_{x(0)}^{x(t)} \frac{dx}{x} = \int_0^t \lambda dt$$

$$\rightarrow \ln[x(t)] - \ln[x(0)] = \lambda t$$

$$\rightarrow x(t) = x(0)e^{\lambda t} \qquad (3.7\text{-}4)$$

the solution to which is

DISTANZA DELLA TRAIETTORIA DAL PUNTO FISSO X_0

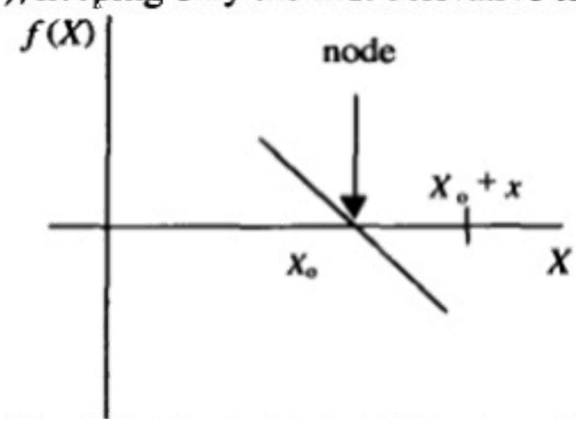
$$\boxed{x(t) = x(0)e^{\lambda t}}$$

where

ESPONENTE DI LYAPUNOV

$$\lambda \equiv \left.\frac{df(X)}{dX}\right|_{X_o}$$

that is, λ is the characteristic value of the fixed point. We see that the trajectory approaches the fixed point (a node) exponentially if $\lambda < 0$ and is repelled from the fixed point (a repellor) exponentially if $\lambda > 0$. λ is also called the ***Lyapunov exponent*** for the region around the fixed point. We should emphasize that these results hold only in the immediate neighborhood of the fixed point where the Taylor series expansion Eq. (3.7-1), keeping only the first derivative term, is a good description of the function $f(X)$.



3.8 Trajectories in a One-Dimensional State Space

What kinds of trajectories can we have in a one-dimensional state space? First, we should note that our analysis thus far simply tells us what happens in the neighborhood of fixed points or what we might call the ***local*** behavior of the system. As we have seen, this local behavior is determined by the nature of the derivatives of the time evolution function evaluated at the fixed point. To obtain a larger-scale picture of the trajectories (a so-called ***global*** picture or a ***global phase portrait***), we need to consider the relationship between the positions of different kinds of fixed points.

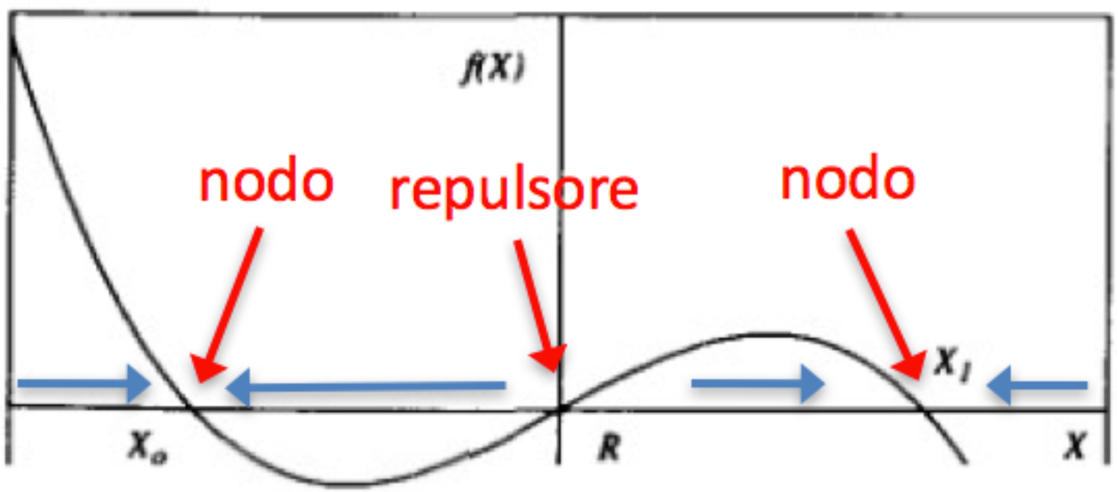


Fig. 3.6. In a one-dimensional state space, two nodes (here labeled X_0 and X_1) must have a repellor R located between them.

L'Equazione Logistica (o di Verhulst)

P.F.Verhulst
(1804-1849)

Exercise 3.8-3. The logistic differential equation. The following differential equation has a "force" term that is identical to the logistic map function introduced in Chapter 1

$$\dot{X} = AX(1-X) \qquad A \in [0,1]$$

(a) Find the fixed points for this differential equation.
(b) Determine the characteristic value and type of each of the fixed points.

Mappa Logistica

$$\boxed{x_{n+1} = Ax_n(1-x_n)}$$

Modello di crescita

Avendo supposto che il numero di individui di una popolazione sia una funzione continua del tempo $N(t)$ che ammette derivata continua, si ha che l'incremento della popolazione al variare del tempo può essere rappresentato dalla derivata di $N(t)$, che in un modello elementare si può supporre direttamente proporzionale al numero di individui della popolazione stessa.

Si ha pertanto la seguente equazione differenziale:

$$\frac{d}{dt}N = rN(t) \quad \rightarrow N(t) = N_o e^{rt}$$ *Crescita Maltusiana (esponenziale)*

con r: parametro di crescita malthusiana (tasso massimo di crescita della popolazione).

Pertanto se r è una costante la popolazione cresce in maniera esponenziale con pendenza dipendente da r.

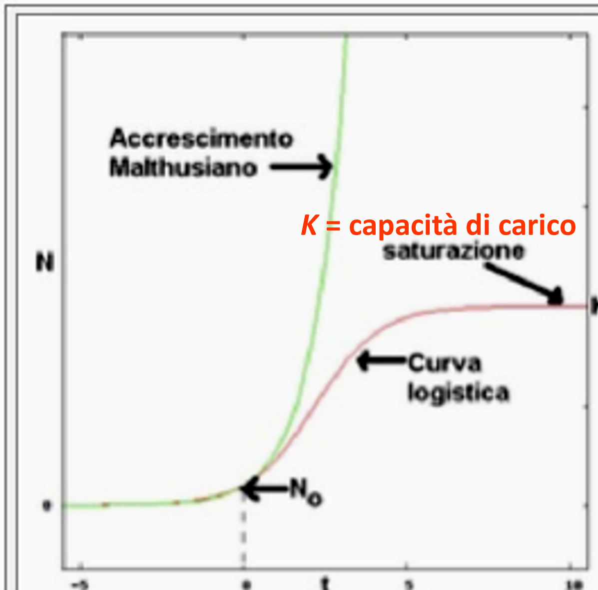
Invece in un ambiente la cui disponibilità di risorse è limitata si può descrivere l'evoluzione della popolazione utilizzando un coefficiente r che decresce all'aumentare della popolazione: il modello più semplice è $r(t)$: $= a - bN(t)$ con a e b costanti. Sostituendo tale funzione nella precedente equazione differenziale si ottiene:

$$\frac{dN}{dt} = aN(t) - bN^2(t)$$

che può essere posta nella forma:

$$\boxed{\frac{dN}{dt} = aN\left(1 - \frac{N}{K}\right)} \quad \xleftrightarrow{\text{se a=b (K=1)}} \quad \boxed{\dot{N}(t) = aN(1-N)}$$

con $K = \frac{a}{b}$ che è la cosiddetta popolazione massima sostenibile ed a uguale al parametro di crescita malthusiana.



Confronto tra curva logistica e curva di accrescimento esponenziale (malthusiano). I parametri sono:
$k = 10, N_0 = 1, r = 1$