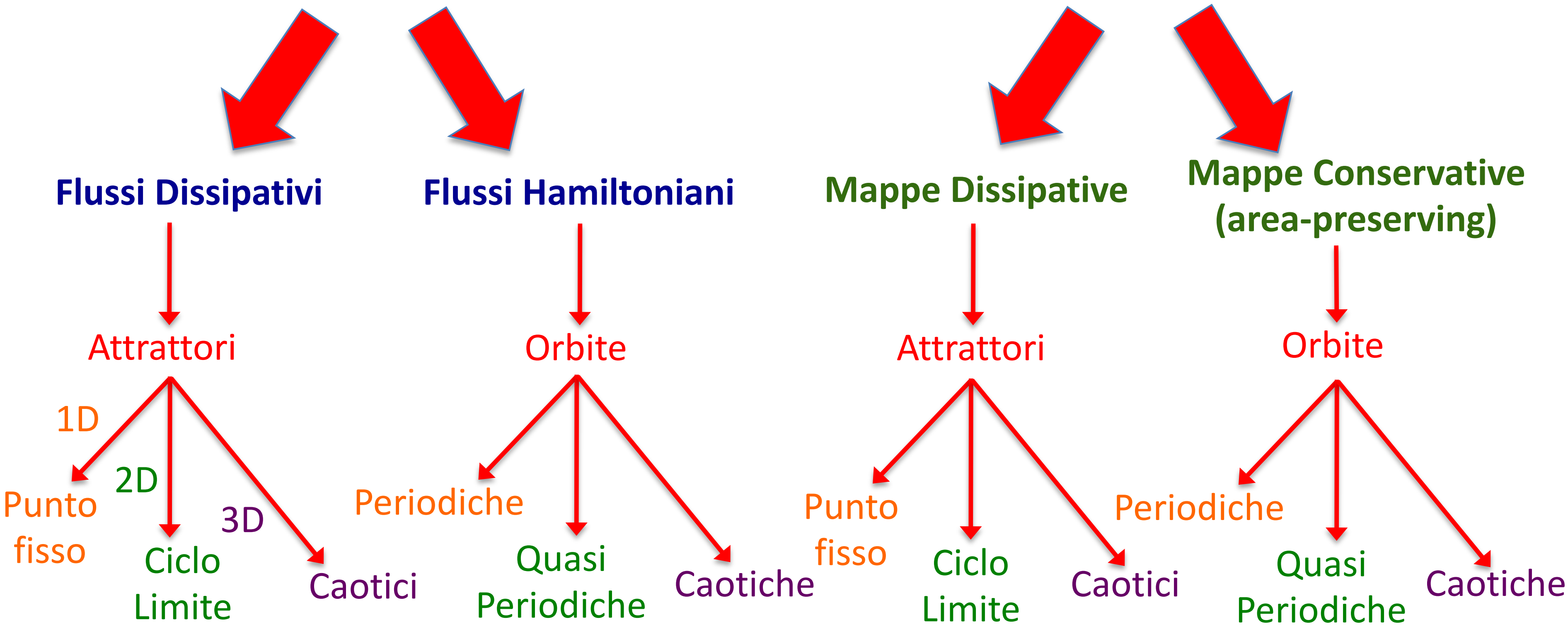
Classificazione dei Sistemi Dinamici

Sistemi dinamici continui (Flussi)

$$\dot{X} = f(X)$$

Sistemi dinamici discreti (Mappe)

$$x_{n+1} = A x_n (1 - x_n) \equiv f_A(x)$$



Classificazione dei Sistemi Dinamici

Sistemi dinamici continui (Flussi)

$$\dot{X} = f(X)$$

Flussi Dissipativi

Attrattori

- 1D: Punto fisso
- 2D: Ciclo Limite
- 3D: Caotici

Flussi Hamiltoniani

Orbite

- Periodiche
- Quasi Periodiche
- Caotiche

Sistemi dinamici discreti (Mappe)

$$x_{n+1} = Ax_n(1-x_n) \equiv f_A(x)$$

Mappe Dissipative

Attrattori

- Punto fisso
- Ciclo Limite
- Caotici

Mappe Conservative (area-preserving)

Orbite

- Periodiche
- Quasi Periodiche
- Caotiche

$$\boxed{\dot{X} = f(X)}$$

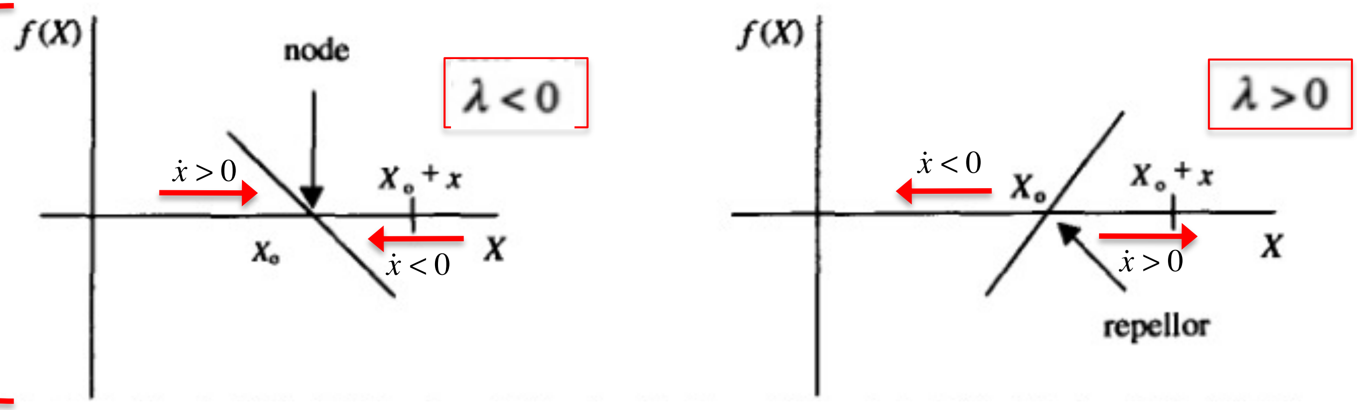
Flussi dissipativi in una dimensione

fixed points (dim.0)

Riepilogo dei Punti Fissi in uno Spazio degli Stati a Una Dimensione

$$\dot{X}\Big|_{X=X_o} = f(X_o) = 0$$

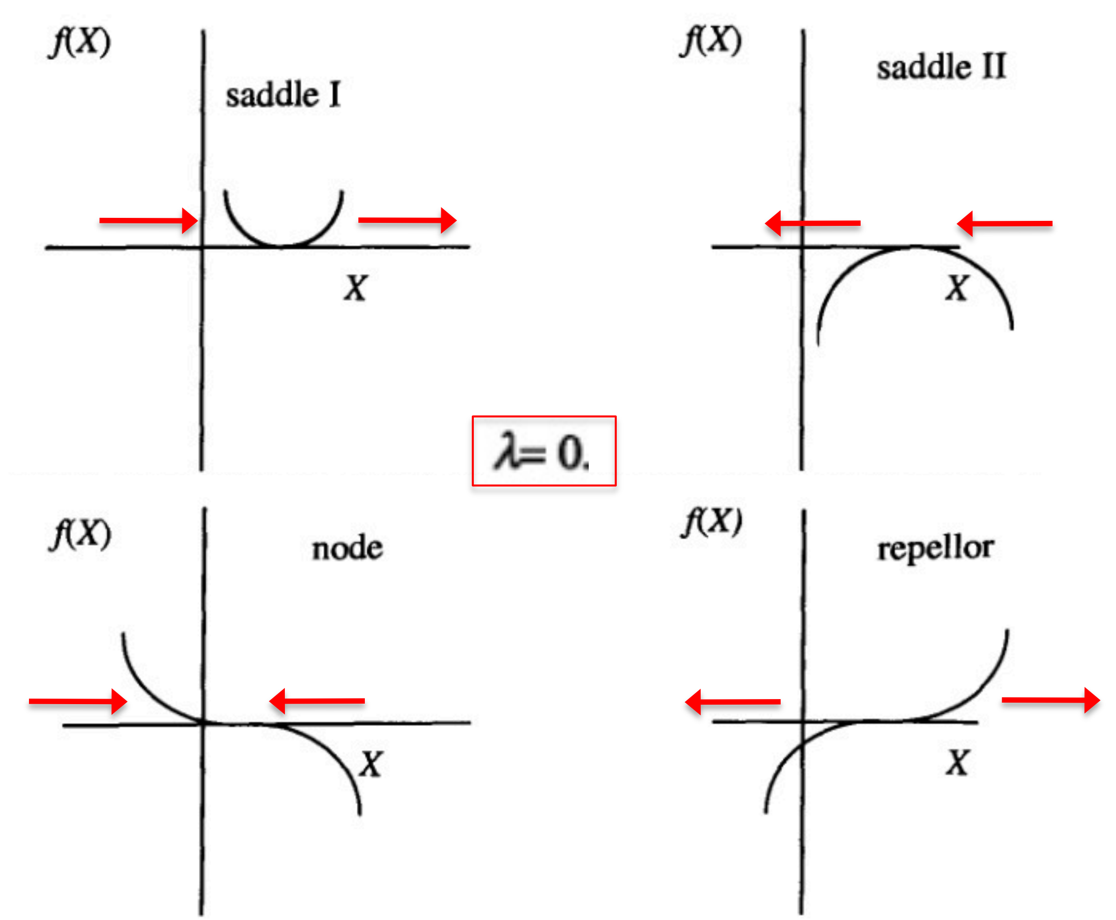
Punti Fissi Strutturalmente Stabili



$$\lambda = \frac{df(X)}{dX}\Big|_{X=X_o}$$

Valore Caratteristico del Punto Fisso

Punti Fissi Strutturalmente Instabili



$$\dot{x} = \frac{df}{dX}\Big|_{X_o} x$$

Equazione linearizzata

$$x(t) = x(0)e^{\lambda t}$$

L'Equazione Logistica (o di Verhulst)

P.F.Verhulst
(1804-1849)

Exercise 3.8-3. The logistic differential equation. The following differential equation has a "force" term that is identical to the logistic map function introduced in Chapter 1

$$\dot{X} = AX(1-X) \qquad A \in [0,1]$$

(a) Find the fixed points for this differential equation.
(b) Determine the characteristic value and type of each of the fixed points.

Mappa Logistica

$$\boxed{x_{n+1} = Ax_n(1-x_n)}$$

Modello di crescita

Avendo supposto che il numero di individui di una popolazione sia una funzione continua del tempo $N(t)$ che ammette derivata continua, si ha che l'incremento della popolazione al variare del tempo può essere rappresentato dalla derivata di $N(t)$, che in un modello elementare si può supporre direttamente proporzionale al numero di individui della popolazione stessa.

Si ha pertanto la seguente equazione differenziale:

$$\frac{d}{dt}N = rN(t) \quad \rightarrow N(t) = N_o e^{rt}$$ ***Crescita Maltusiana (esponenziale)***

con r: parametro di crescita malthusiana (tasso massimo di crescita della popolazione).

Pertanto se r è una costante la popolazione cresce in maniera esponenziale con pendenza dipendente da r.

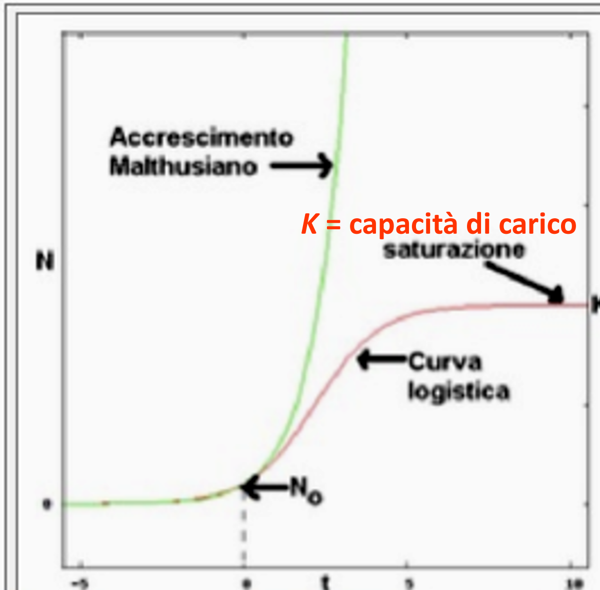
Invece in un ambiente la cui disponibilità di risorse è limitata si può descrivere l'evoluzione della popolazione utilizzando un coefficiente r che decresce all'aumentare della popolazione: il modello più semplice è $r(t)$: $= a - bN(t)$ con a e b costanti. Sostituendo tale funzione nella precedente equazione differenziale si ottiene:

$$\frac{dN}{dt} = aN(t) - bN^2(t)$$

che può essere posta nella forma:

$$\boxed{\frac{dN}{dt} = aN\left(1 - \frac{N}{K}\right)} \quad \xleftrightarrow{\text{se a=b (K=1)}} \quad \boxed{\dot{N}(t) = aN(1-N)}$$

con $K = \frac{a}{b}$ che è la cosiddetta popolazione massima sostenibile ed a uguale al parametro di crescita malthusiana.



Confronto tra curva logistica e curva di accrescimento esponenziale (malthusiano). I parametri sono: $k = 10, N_0 = 1, r = 1$

Studio dei punti fissi dell'Equazione Logistica

Exercise 3.8-3. The logistic differential equation. The following differential equation has a "force" term that is identical to the logistic map function introduced in Chapter 1

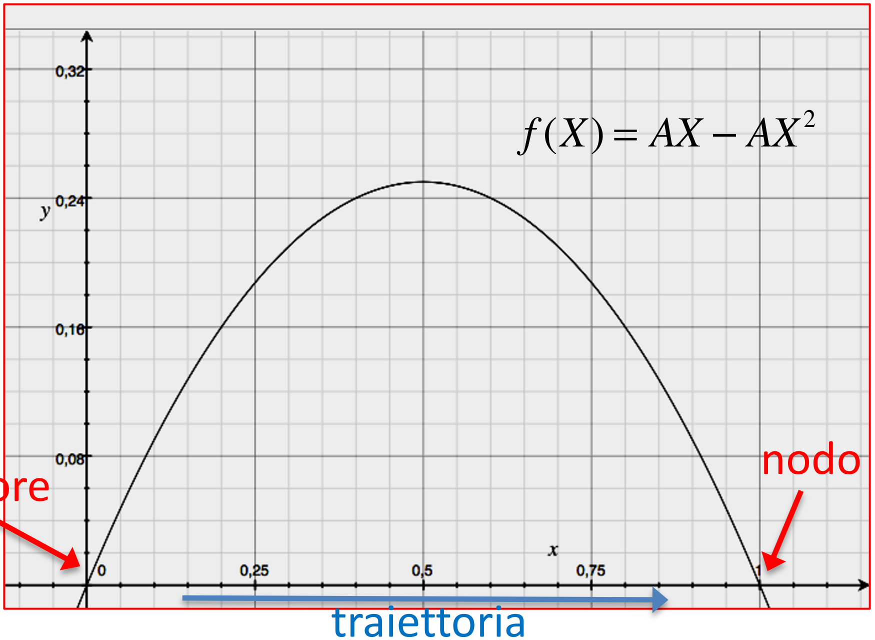
$$\dot{X} = AX(1-X) \qquad A \in [0,1]$$

(a) Find the fixed points for this differential equation.
(b) Determine the characteristic value and type of each of the fixed points.

a) $\dot{X}\big|_{X=X_0} = 0 \rightarrow AX_0(1-X_0) = 0 \rightarrow$

Punti Fissi

$X_0 = 0\,,\ X_0 = 1$



b) $\lambda = \dfrac{df(X)}{dX}\bigg|_{X=X_0}$

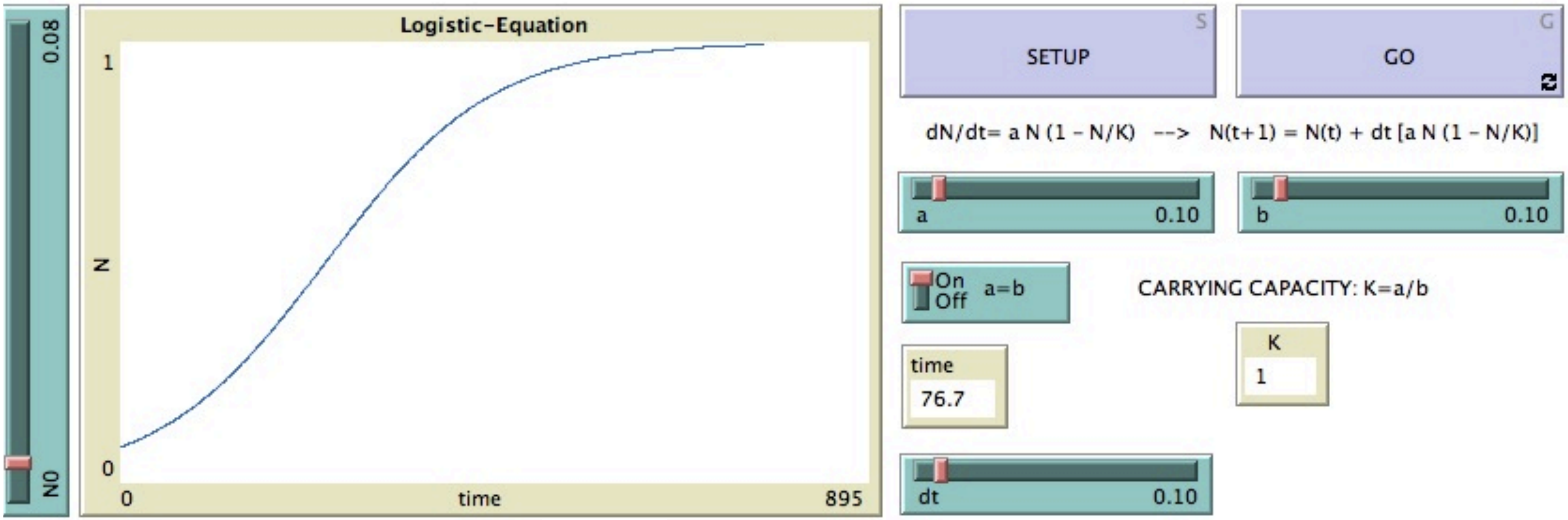
$$f(X) = AX - AX^2 \quad \rightarrow \quad \frac{df(X)}{dX} = A - 2AX$$

$$\rightarrow \begin{cases} \lambda(X_0 = 0) = A > 0 \\ \lambda(X_0 = 1) = A - 2A = -A < 0 \end{cases}$$

→ X_0=0 è un repulsore (punto fisso repulsivo - instabile)

→ X_0=1 è un nodo (punto fisso attrattivo - stabile)

equazione_logistica.nlogo



METODO DI INTEGRAZIONE DI EULERO

Si supponga di voler approssimare la soluzione del problema di Cauchy:

$$y'(t) = f(t, y(t)) \qquad y(t_0) = y_0$$

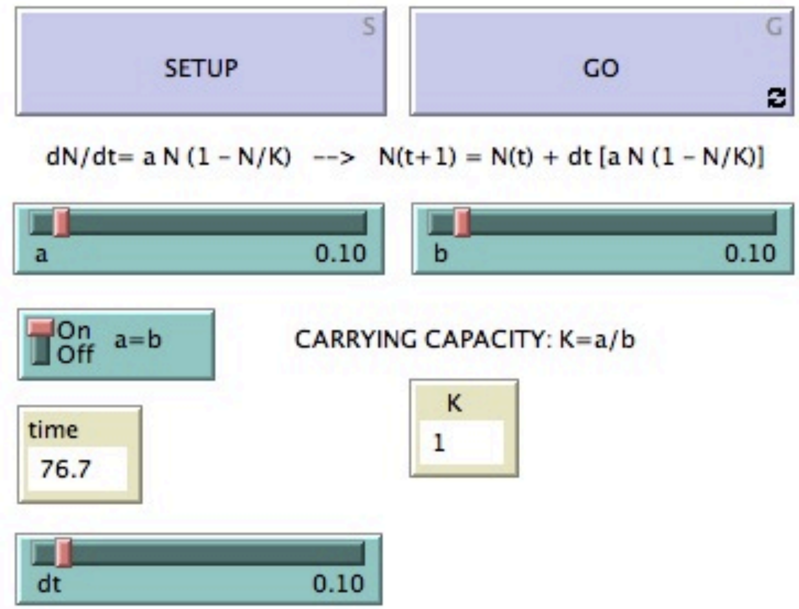
discretizzando la variabile t, quindi definendo $t_n = t_0 + nh$, con h la dimensione di ogni intervallo. Tra t_n e $t_{n+1} = t_n + h$ il comportamento della soluzione può essere approssimato stimando:

$$(y_{n+1} - y_n)/h = f(t_n, y_n) \quad \rightarrow \quad y_{n+1} = y_n + hf(t_n, y_n)$$

dove il valore di $y_n \approx y(t_n)$ risulta essere un'approssimazione della soluzione della ODE al tempo t_n. Il metodo di Eulero è esplicito, ovvero la soluzione y_{n+1} è una funzione esplicita di y_i per $i \leq n$.

equazione_logistica.nlogo

```
globals [K N time cont]

to setup

  ca
  set-current-plot "Logistic-Equation"
  set-plot-x-range 0 10
  set-plot-y-range 0 1.2

  if (a=b) [set b a]
  set K (a / b)
  set N N0
  set cont 0

  do-plot

end

to go

 set time (cont * dt)

 ;Integrazione con Eulero (primo ordine)
 set N  (N + (dt * (a * N *(1 - (N / K)))))

 do-plot
 wait 0.01

 set cont cont + 1

end

to do-plot
  set-current-plot-pen "N(t)"
  plotxy time N
  set-current-plot-pen "K"
  plotxy time K
end
```



METODO DI INTEGRAZIONE DI EULERO

3.9 Dissipation Revisited in 1D

$$\boxed{\dot{X} = f(X)}$$

Earlier in this chapter, we mentioned that we would be interested primarily in dissipative systems. How do we know if a particular system, here represented by a particular function $f(X)$, is dissipative or not? If we are modeling a real physical system, the dissipation is due to friction (in the generalized sense), viscosity, and so on, and usually we can decide on physical grounds whether or not dissipation is important. However, it would be useful to have a mathematical tool that we could use to recognize a dissipative system directly from its dynamical equations. Given this tool we could check to see if a mathematical model we have developed (or which someone has given to us) includes dissipation or not.

To assess dissipation, we will use an important conceptual tool: a "cluster" of initial conditions. In the one-dimensional case, the cluster of initial conditions is some (relatively) small segment of the X axis. (We exclude segments that contain fixed points for what will become obvious reasons.) Let us suppose that this line segment runs from X_A to X_B (with $X_B > X_A$). See Fig. 3.7. The length of the segment is $X_B - X_A$. We want to examine what happens to the length of this line segment as time evolves and the trajectory points in that segment move through the state space. The time rate of change of the length of this segment is given by

$$\frac{d}{dt}(X_B - X_A) = \dot{X}_B - \dot{X}_A = f(X_B) - f(X_A) \tag{3.9-1}$$

"velocity" of X_A "velocity" of X_B

Fig. 3.7. A "cluster of initial conditions," indicated by the heavy line, along the X axis.

Thus, if $f(X_B) < f(X_A)$, the length of the segment will shrink as time goes on. If the line segment is sufficiently short, we can use the Taylor series expansion

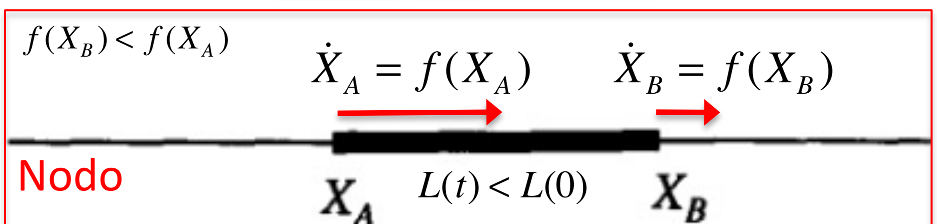
$$f(X_B) = f(X_A) + \left.\frac{df}{dX}\right|_{X_A}(X_B - X_A) + \ldots \tag{3.9-2}$$

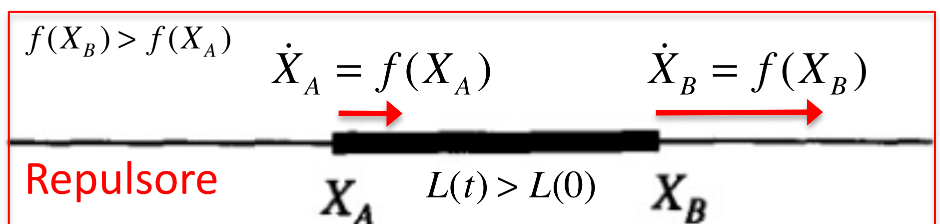
to relate $f(X_B)$ to $f(X_A)$. If we let $L = X_B - X_A$, and keep only the first derivative term in Eq. (3.9-2), then we can write Eq. (3.9-1) in the form

$$\frac{d}{dt}(X_B - X_A) = \dot{X}_B - \dot{X}_A = f(X_B) - f(X_A) \rightarrow \quad \frac{1}{L}\frac{dL}{dt} = \frac{1}{L}\left[f(X_B) - f(X_A)\right] = \frac{df(X)}{dX} \tag{3.9-3}$$

From Eq. (3.9-3), we see that the length of the segment of initial conditions will decrease if $f(X_B) < f(X_A)$ or, equivalently, if df/dX is negative. This condition will be satisfied if the trajectories are approaching a node, since the derivative of f is negative at a node and, by continuity, in the neighborhood of a node. (We are excluding the structurally unstable fixed points from our consideration.)

The previous analysis concentrated on the behavior near a single fixed point. More generally, we can ask for the "average" behavior over the history of some trajectory. It may turn out that a cluster of initial conditions first expands, as it leaves the region around a repellor, and then later contracts as it approaches a node. On the average, the cluster of trajectory points must experience contraction for a bounded dissipative system.





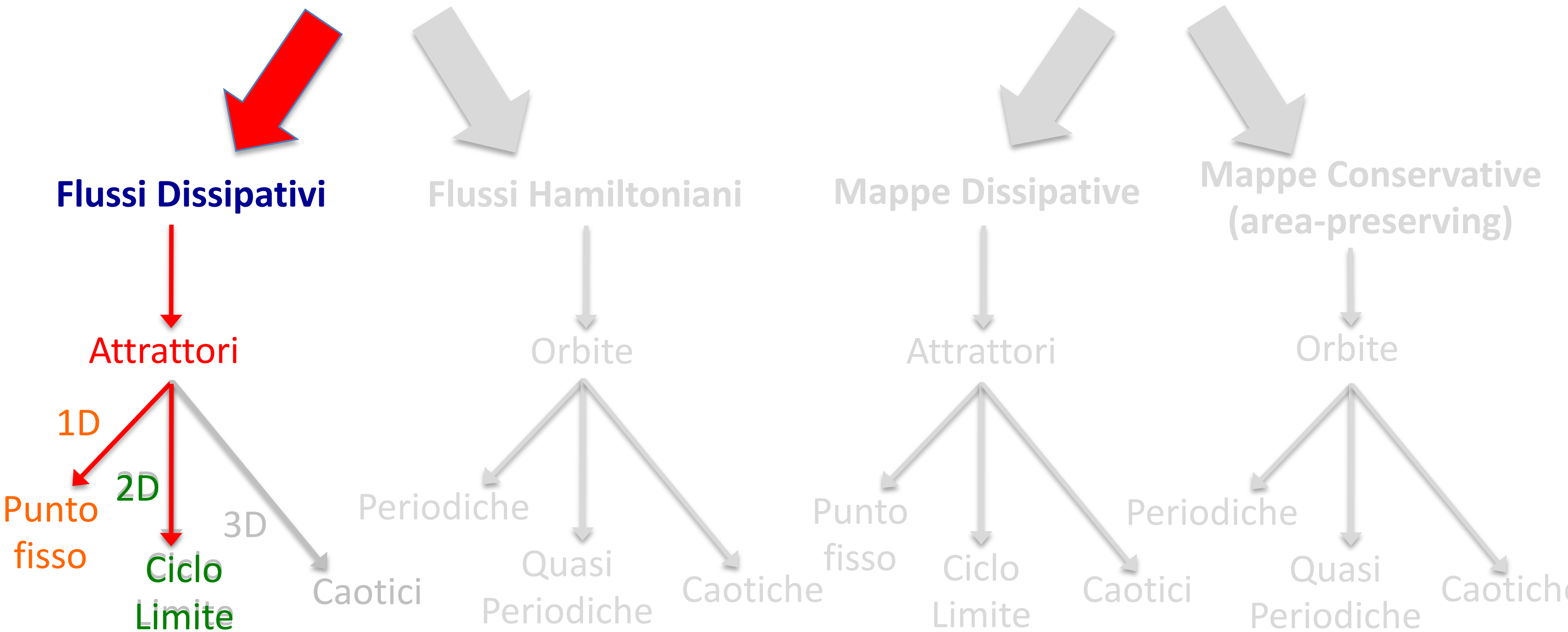
Classificazione dei Sistemi Dinamici

Sistemi dinamici continui (Flussi)

$$\dot{X} = f(X)$$

Sistemi dinamici discreti (Mappe)

$$x_{n+1} = Ax_n(1 - x_n) \equiv f_A(x)$$



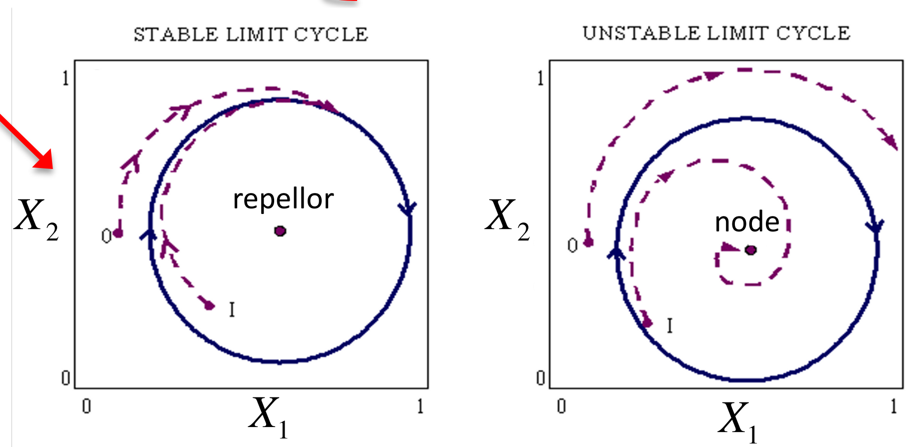
3.10 Two-Dimensional State Space

We now extend our discussion of state space to two-dimensional systems, where we shall see that the greater freedom provided by the higher dimensionality increases significantly the variety of behaviors and at the same time lifts some, but not all, of the geometrical constraints on the pattern of fixed points. Also, we shall see that a new type of attractor, a ***limit cycle***, must be introduced to describe some of these new types of behavior.

Our discussion for two-dimensional state spaces will proceed along the same lines as the discussion of one-dimensional systems. We assume that the equations describing the dynamics of the system can be written as a pair of coupled, first-order differential equations for the state variables, which we shall label X_1 and X_2. (Occasionally, we will use x and y as the independent variables, but we want to emphasize that in general the state space variables are not spatial coordinate variables.) The time evolution equations are

$$\begin{cases} \dot{X}_1 = f_1(X_1, X_2) \\ \dot{X}_2 = f_2(X_1, X_2) \end{cases} \qquad (3.10\text{-}1)$$

SPAZIO DEGLI STATI 2D



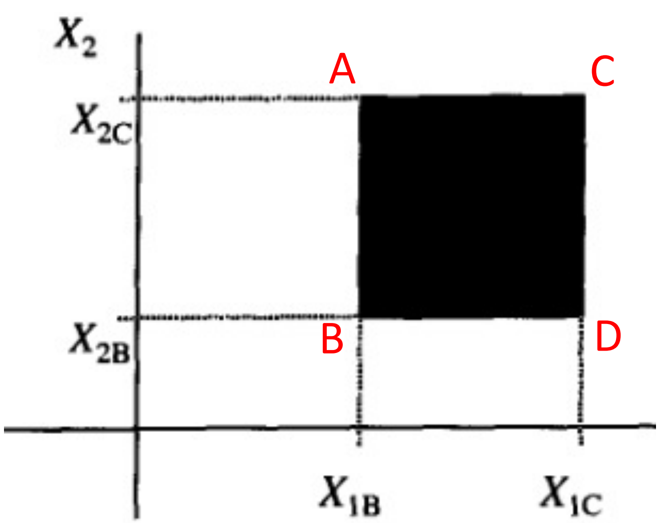
3.13 Dissipation and the Divergence Theorem

Now we show how we can test for dissipation in two-dimensional state space. We shall then see that, in principle, the generalization to many dimensions is easy. In two dimensions, we start with a cluster of initial conditions of the two variables X_1 and X_2 in some (small) area delimited by the coordinates (X_{1C}, X_{2C}) and (X_{1B}, X_{2B}) as shown in Fig. 3.11.

Again we compute the rate of change of that area

$$A = (X_{1C} - X_{1B})(X_{2C} - X_{2B}) \tag{3.13-1}$$

$$\begin{aligned}\frac{dA}{dt} &= (X_{1C} - X_{1B})\{f_2(\overset{A}{X_{1B}, X_{2C}}) - f_2(\overset{B}{X_{1B}, X_{2B}})\} \\ &+ \{f_1(\underset{D}{X_{1C}, X_{2B}}) - f_1(\underset{B}{X_{1B}, X_{2B}})\}(X_{2C} - X_{2B})\end{aligned} \tag{3.13-2}$$



where we have used the time-evolution equations

$$\dot{X}_1 = f_1(X_1, X_2)$$
$$\dot{X}_2 = f_2(X_1, X_2)$$

Fig. 3.11. A rectangle of initial conditions in state space of two variables X_1 and X_2.

$$\frac{dA}{dt} = (X_{1C} - X_{1B})\{\overset{A}{f_2(X_{1B}, X_{2C})} - \overset{B}{f_2(X_{1B}, X_{2B})}\} + \{\underset{D}{f_1(X_{1C}, X_{2B})} - \underset{B}{f_1(X_{1B}, X_{2B})}\}(X_{2C} - X_{2B}) \tag{3.13-2}$$

We make use of a Taylor series expansion

$$\underset{D}{f_1(X_{1C}, X_{2B})} = \underset{B}{f_1(X_{1B}, X_{2B})} + (X_{1C} - X_{1B}) \left.\frac{\partial f_1}{\partial X_1}\right|_{X_{1B}, X_{2B}} + \ldots \tag{3.13-4}$$

with a similar expression for f_2. When these expansions are substituted into Eq. (3.13-2), we obtain, after dividing through by A

$$\boxed{\frac{1}{A}\frac{dA}{dt} = \frac{\partial f_1}{\partial X_1} + \frac{\partial f_2}{\partial X_2}} = (f_{11} + f_{22}) \tag{3.13-5}$$

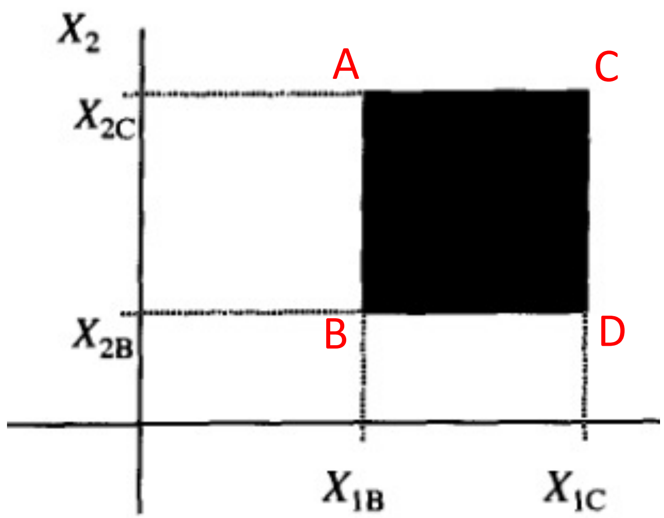


Fig. 3.11. A rectangle of initial conditions in state space of two variables X_1 and X_2.

Per 2 dimensioni: $$\frac{1}{A}\frac{dA}{dt} = \frac{\partial f_1}{\partial X_1} + \frac{\partial f_2}{\partial X_2} < 0 \qquad (3.13\text{-}5)$$

Once again we see that the relative growth or shrinkage of the area containing the set of initial conditions is determined by the derivatives (here partial derivatives) of the time evolution functions. If the right-hand side of Eq. (3.13-5) is negative, then the initial phase space area shrinks to 0, and we say that the system is dissipative. The trajectories all collapse to an attractor whose geometric dimension is less than that of the original state space. For two state space dimensions, the attractor could be a point (a node) or a curve (a limit cycle). It should be (almost) obvious that for N dimensions, the evolution of an N-dimensional volume V of initial conditions in state space is given by

Per N dimensioni: $$\boxed{\frac{1}{V}\frac{dV}{dt} = \sum_{i=1}^{N}\frac{\partial f_i}{\partial x_i} \equiv div(f) < 0} \qquad (3.13\text{-}6)$$

where the right-hand equality defines what is called the ***divergence*** of the set of functions f_i. If $div(f) < 0$ on the average over state space, we know that the initial volume of initial conditions will collapse onto a geometric region whose dimensionality is less than that of the original state space, and we know that the state space has at least one attractor.

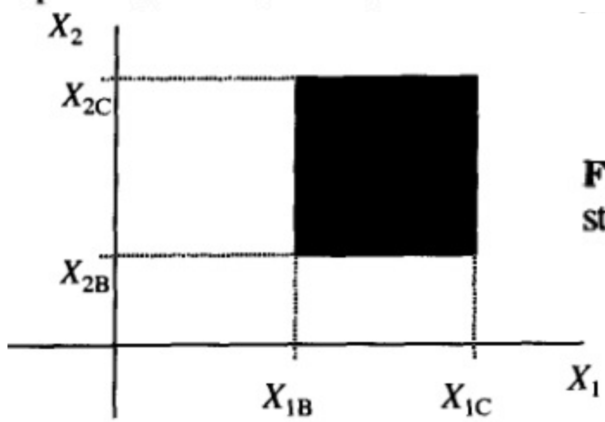


Fig. 3.11. A rectangle of initial conditions in state space of two variables X_1 and X_2.

$$\boxed{\begin{aligned}\dot{X}_1 &= f_1(X_1.X_2)\\ \dot{X}_2 &= f_2(X_1.X_2)\end{aligned}}$$

Flussi dissipativi in due dimensioni

$$\frac{1}{A}\frac{dA}{dt} = \frac{\partial f_1}{\partial X_1} + \frac{\partial f_2}{\partial X_2} \quad < 0$$

fixed points (dim.0)

limit cycles (dim.1)

3.10 Two-Dimensional State Space

$$\begin{cases} \dot{X}_1 = f_1(X_1, X_2) \\ \dot{X}_2 = f_2(X_1, X_2) \end{cases} \tag{3.10-1}$$

The behavior of the system is followed by looking at trajectories in an X_1-X_2 state space. Just as in one-dimension, the fixed points of Eq. (3.10-1) play a major role in the dynamics of the system. The fixed points, of course, are those points (X_{1o}, X_{2o}) satisfying

$$\begin{cases} f_1(X_{1o}, X_{2o}) = 0 \\ f_2(X_{1o}, X_{2o}) = 0 \end{cases} \tag{3.10-2}$$

You have probably already anticipated the next step: The character of the fixed point and the behavior of trajectories in the neighborhood of the fixed point are determined by the derivatives of the functions f_1 and f_2 evaluated at the fixed point; however, since f_1 and f_2 generally depend on both X_1 and X_2, there are four partial derivatives to consider

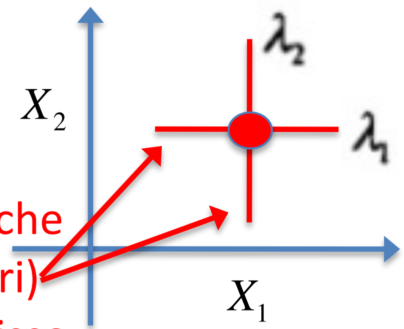
$$\frac{\partial f_1}{\partial X_1}, \quad \frac{\partial f_1}{\partial X_2}, \quad \frac{\partial f_2}{\partial X_1}, \quad \frac{\partial f_2}{\partial X_2} \tag{3.10-3}$$

The question then is how the characteristics of the fixed point depend on those four partial derivatives.

A Special Case

Before considering the general problem of fixed point characteristics in two dimensions, let us first look at a particularly simple case—the case for which only two of the four derivatives are not equal to 0. In particular, let us assume that at the fixed point (X_{1o}, X_{2o}) the derivatives have the following values:

Direzioni Caratteristiche (autovettori) del Punto Fisso



$$\boxed{\frac{\partial f_1}{\partial X_1} = \lambda_1} \qquad \frac{\partial f_1}{\partial X_2} = 0$$

$$\frac{\partial f_2}{\partial X_1} = 0 \qquad \boxed{\frac{\partial f_2}{\partial X_2} = \lambda_2} \tag{3.10-4}$$

Valori Caratteristici (autovalori) del Punto Fisso

In this special case, what happens along the X_1 direction in the neighborhood of the fixed point depends only on λ_1, and what happens along the X_2 direction depends only on λ_2. For this case, we say that the X_1 and X_2 axes are the ***characteristic directions*** with the associated characteristic values λ_1 and λ_2. (Please keep in mind that this independence of the X_1 and X_2 motions holds only in this special case and only in the vicinity of this fixed point.)

Types of Fixed Points in Two Dimensions

We can now begin to construct the catalog of types of fixed points in two dimensions by fitting together the possible types of one-dimensional behavior. We shall soon see, however, that there are new types of behavior possible in two dimensions. In the simplest case, λ_1 and λ_2 are both real numbers and both are nonzero. (When a characteristic value equals 0, then we need a more complicated analysis, just as we did in one-dimension.) By using arguments like those leading up to Eq. (3.6-3), we can see that there are four possible fixed points as listed in Table 3.2. In Fig. 3.8, sample trajectories are shown in the neighborhood of those fixed points.

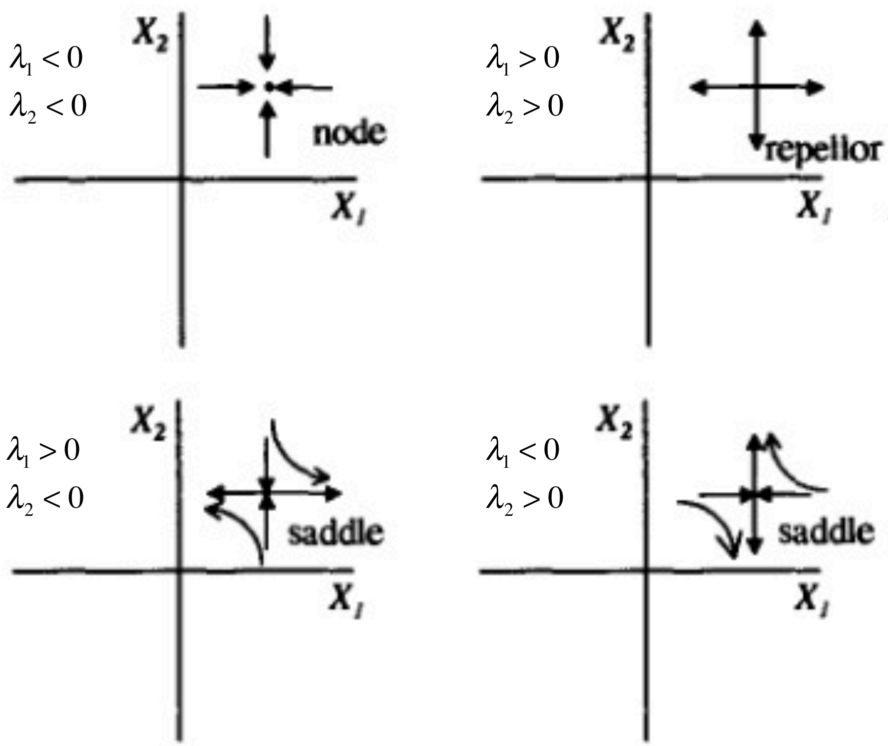


Fig. 3.8. Sample trajectories near each of the four types of fixed points with real characteristic values in two dimensions.

Table 3.2.
Possible Fixed Point Characters with Real Characteristic Values

λ_1	λ_2	Type of Fixed Point
< 0	< 0	attracting node
> 0	> 0	repellor
> 0	< 0	saddle point
< 0	> 0	saddle point

We are now in a position to understand why a saddle point is called a saddle point. The behavior of trajectories near a saddle point is analogous to the behavior of a ball rolling under the influence of gravity on a saddle-shape surface as shown in Fig. 3.9. In that picture, a ball rolling along the x axis will be attracted to the saddle point at (0,0). A ball rolling along the y axis will roll away from (be "repelled by") the saddle point.

In more formal terms the connection is made by defining a function $g(x,y)$ (to use the variables indicated in Fig. 3.9) such that

$$\dot{x} = f_1(x,y) \qquad\qquad \left\{ \begin{aligned} f_1(x,y) &= -\frac{\partial g(x,y)}{\partial x} \\ f_2(x,y) &= -\frac{\partial g(x,y)}{\partial y} \end{aligned} \right. \qquad (3.10\text{-}5)$$
$$\dot{y} = f_2(x,y)$$

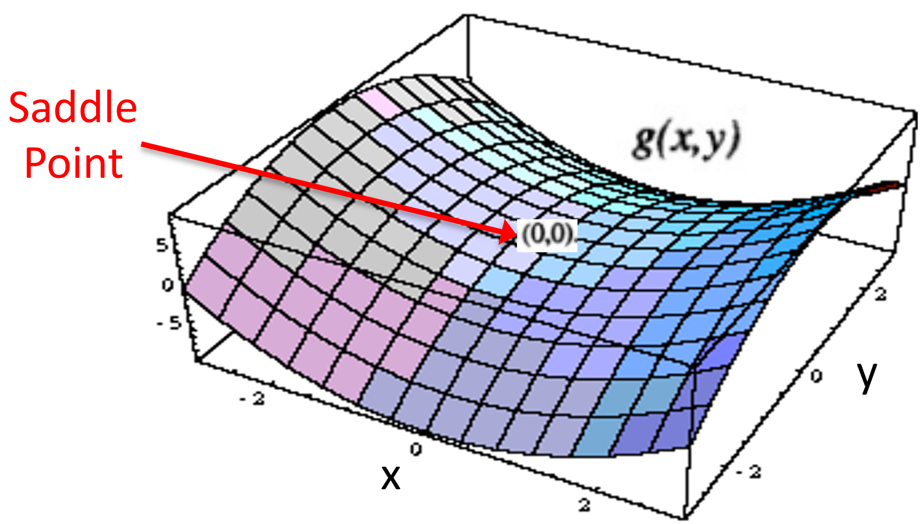


Fig. 3.9. A saddle-type surface for a two-dimensional state space. The saddle point is located at $(x,y) = (0,0)$.

at a fixed point $P_0(x_0, y_0)$

$$\begin{cases} f_1(x,y) = -\dfrac{\partial g(x,y)}{\partial x} = 0 \\ f_2(x,y) = -\dfrac{\partial g(x,y)}{\partial y} = 0 \end{cases} \qquad (3.10\text{-}5)$$

The "force functions" f_1 and f_2 are given by the negative gradients of the "potential function" $g(x,y)$. Then at a fixed point of the f_1, f_2 system the function g has an extremum (a local maximum or minimum). At a saddle point, the function g, as shown in Fig. 3.9, has a minimum while moving along the x axis but a maximum while moving along the y axis. For a mechanical system the function $g(x,y)$ might represent the potential energy function for the system.

Potential Surface

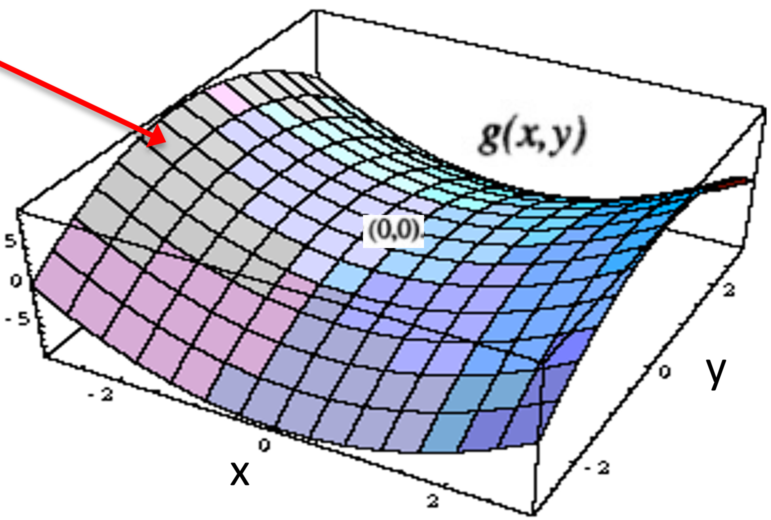


Fig. 3.9. A saddle-type surface for a two-dimensional state space. The saddle point is located at $(x,y) = (0,0)$.

Valori Caratteristici del Saddle Point

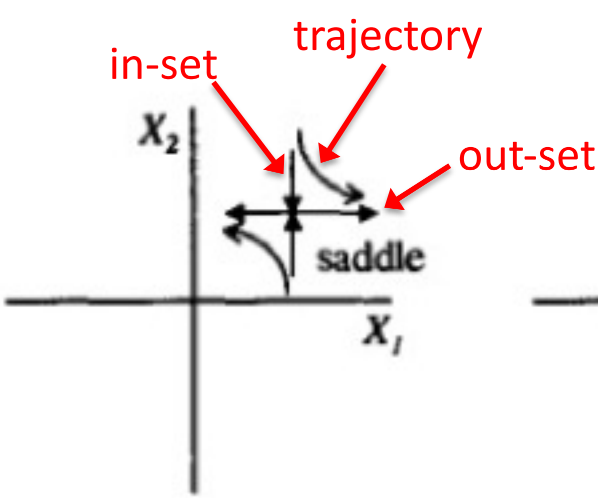
$$\lambda_1 = \left.\frac{\partial f_1(x,y)}{\partial x}\right|_{P_0} = -\left.\frac{\partial^2 g(x,y)}{\partial x^2}\right|_{P_0} < 0$$

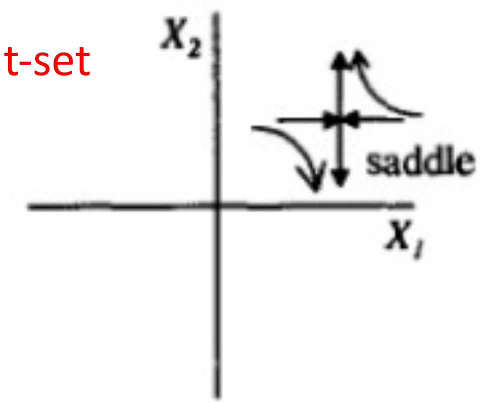
$$\lambda_2 = \left.\frac{\partial f_2(x,y)}{\partial y}\right|_{P_0} = -\left.\frac{\partial^2 g(x,y)}{\partial y^2}\right|_{P_0} > 0$$

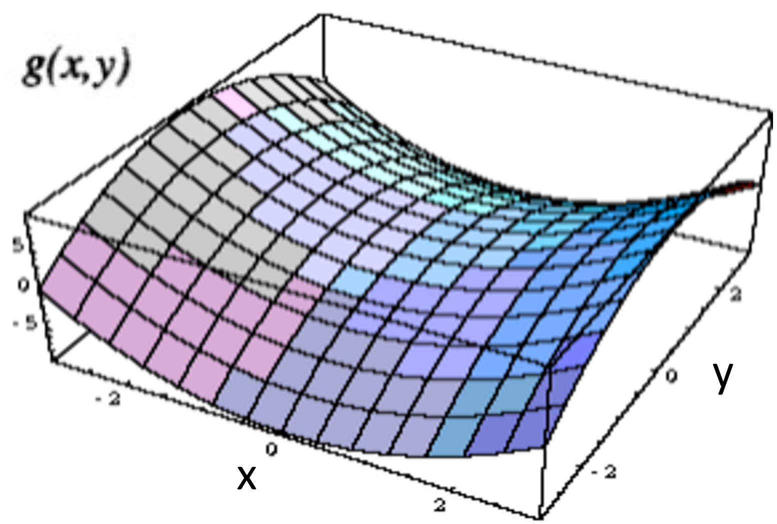
Some Terminology

Saddle points, and in particular the special trajectories that head directly toward or directly away from a saddle point, play an important role, as we shall see, in organizing the behavior of all possible trajectories in state space. Because of this role, special terminology has been developed to talk about these trajectories.

The sets of points that form the trajectories heading directly to (approaching the saddle point as $t \to \infty$) or directly away from a saddle point are sometimes called the ***invariant manifolds*** associated with that saddle point. More specifically, the trajectories heading directly toward the saddle point form what is called the ***stable manifold*** (because the characteristic value $\lambda < 0$ along those trajectories), while the trajectories heading directly away from the saddle point form what is called the ***unstable manifold***. Other authors (e.g. [Abraham and Shaw, 1984] and [Thomson and Stewart, 1986]) call these same manifolds ***insets*** and ***outsets*** respectively. We prefer to call them ***in-sets*** and ***out-sets*** to avoid possible confusion with the usual English meanings of the words inset and outset.



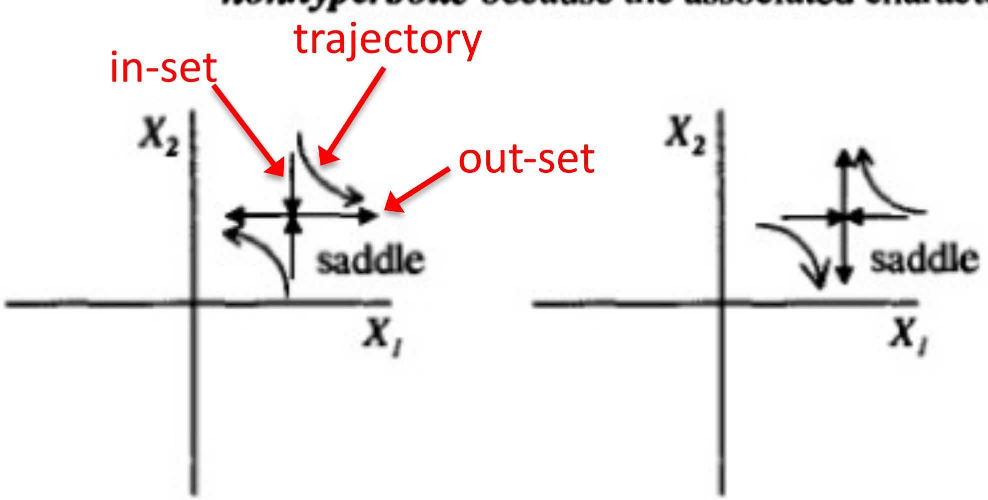


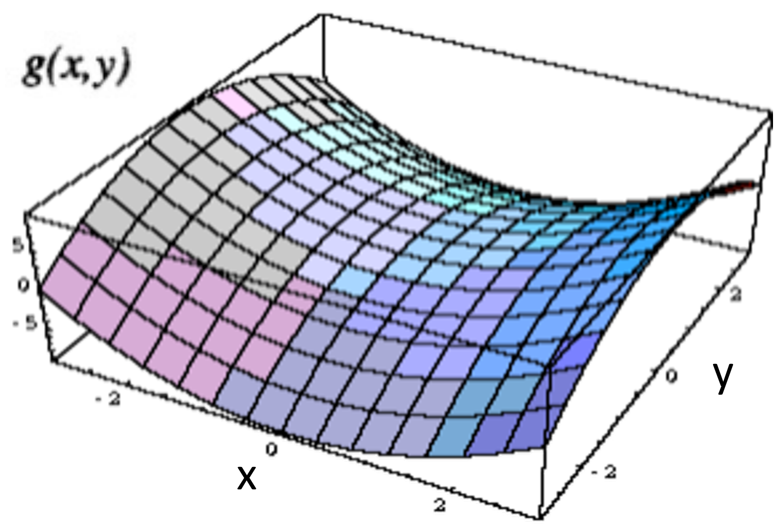


The Importance of Saddle Points

To get a feeling for the importance of saddle points and their in-sets and out-sets, let us consider a system that has only one fixed point. If that fixed point is a saddle point, and if the characteristic values are not equal to zero, then the in-sets and out-sets of that saddle point divide the state space up into four "quadrants." A trajectory that is not an in-set or an out-set is confined to the quadrant in which it starts as illustrated in the lower half of Fig. 3.8. In that sense, the in-sets and out-sets "organize" the state space. The out-sets and in-sets are part of the separatrices (if there are any) for the state space.

For this kind of saddle point (for which neither of the characteristic values is 0), the trajectories near the saddle point but not on either the in-set or out-set look like sections of hyperbolas. Hence, this kind of saddle point is called a ***hyperbolic point***. In fact, the term ***hyperbolic*** is applied to any fixed point whose characteristic values are not equal to 0. (In the general case to be discussed later, the real parts of the characteristic values are not 0.) In this language, the one-dimensional saddle points discussed in the previous section, which we called structurally unstable, are ***nonhyperbolic*** because the associated characteristic value is 0.

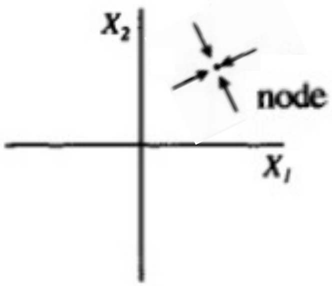




3.11 Two-dimensional State Space: The General Case

$$\frac{\partial f_1}{\partial X_1} \neq 0 \quad \frac{\partial f_1}{\partial X_2} \neq 0$$

$$\frac{\partial f_2}{\partial X_1} \neq 0 \quad \frac{\partial f_2}{\partial X_2} \neq 0$$

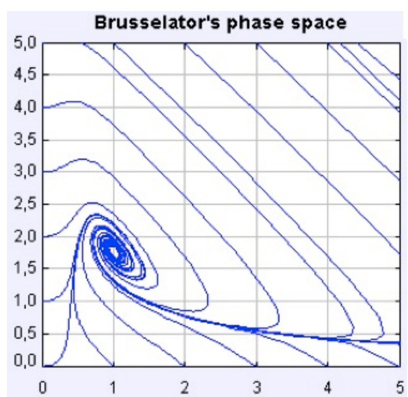


In the most general case in two dimensions, all four of the derivatives in Eq. (3.10-3) are nonzero. How do we characterize the fixed point in that situation? It turns out that in this case there are still just two characteristic values associated with the fixed point, but the associated characteristic directions are no longer the X_1 and X_2 directions, in general.

At this point a specific example will help illustrate these ideas. We will describe the equations used to model a certain set of chemical reactions [Nicolis and Prigogine, 1989], called the Brusselator model because its originators worked in Brussels. The equations are

$$\left\{ \begin{aligned} \dot{X} &= A - (B+1)X + X^2 Y \\ \dot{Y} &= BX - X^2 Y \end{aligned} \right. \tag{3.11-1}$$

A and B are positive numbers that represent the control parameters, and X and Y are variables proportional to the concentrations of some of the intermediate products in the chemical reaction. One can imagine monitoring these concentrations as functions of time with some appropriate electrodes or with some optical absorption measurements that are sensitive to those chemical concentrations.



Ilya Prigogine
(1917-2003)

The Brusselator Model

$$\begin{cases} \dot{X} = A - (B+1)X + X^2 Y \\ \dot{Y} = BX - X^2 Y \end{cases} \qquad (3.11\text{-}1)$$

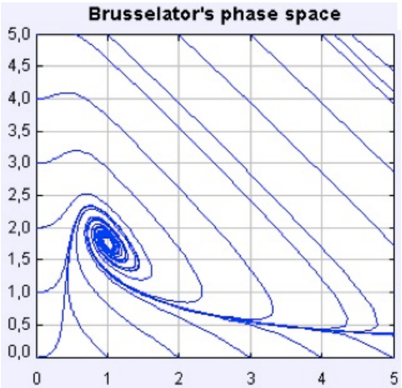
First let us find the fixed points for this set of equations. By setting the time derivatives equal to 0, we find that the fixed points occur at the values X,Y that satisfy

$$\begin{cases} A - (B+1)X + X^2 Y = 0 & (3.11\text{-}2) \\ BX - X^2 Y = 0 & (3.11\text{-}3) \end{cases}$$

We see that there is just one point (X,Y) which satisfies these equations, and the coordinates of that fixed point are $X_0 = A$, $Y_0 = B/A$.

$$\frac{\partial f_1}{\partial X_1} \neq 0 \quad \frac{\partial f_1}{\partial X_2} \neq 0$$

$$\frac{\partial f_2}{\partial X_1} \neq 0 \quad \frac{\partial f_2}{\partial X_2} \neq 0$$



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a-(b+1)x+x^2y=0, bx-x^2y=0

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- crixxi vs dullo
- Solve vs DSolve vs...
- dive 40min at 60ft

Input:

$$\{a-(b+1)\,x+x^2\,y=0,\ b\,x-x^2\,y=0\}$$

Alternate forms:

$$\{a+x^2\,y=(b+1)\,x,\ x^2\,y=b\,x\}$$

$$\{a=(b+1)\,x-x^2\,y,\ b=x\,y\ \bigvee\ x=0\}$$

$$\{a-b\,x+x^2\,y-x=0,\ x\,(b-x\,y)=0\}$$

$e_1 \vee e_2 \vee \ldots$ is the logical OR function »

Alternate form assuming a, b, x and y are positive:

$$\{a+x^2\,y=(b+1)\,x,\ b=x\,y\}$$

Expanded form: Show steps

$$\{a-b\,x+x^2\,y-x=0,\ b\,x-x^2\,y=0\}$$

Solution:

$$x=a\,,\quad a\neq 0\,,\quad y=\frac{b}{a}$$

Punti Fissi in due dimensioni: il caso generale

$$\dot{X}_1 = f_1(X_1.X_2)$$
$$\dot{X}_2 = f_2(X_1.X_2) \qquad (3.10\text{-}1)$$

What is the character of that fixed point? To see how we find these characteristic values, let us return to our general two-dimensional state space and make use of a Taylor series expansion of Eq. (3.10-1) in the neighborhood of the fixed points (X_{1o}, X_{2o}):

$$\dot{X}_1 = f_1(X_1, X_2) = (X_1 - X_{1o})\frac{\partial f_1}{\partial X_1} + (X_2 - X_{2o})\frac{\partial f_1}{\partial X_2} + \ldots \qquad (3.11\text{-}4a)$$

DISTANZA DELLA TRAIETTORIA DAL PUNTO FISSO LUNGO L'ASSE X

DISTANZA DELLA TRAIETTORIA DAL PUNTO FISSO LUNGO L'ASSE Y

$$\dot{X}_2 = f_2(X_1, X_2) = (X_1 - X_{1o})\frac{\partial f_2}{\partial X_1} + (X_2 - X_{2o})\frac{\partial f_2}{\partial X_2} + \ldots \qquad (3.11\text{-}4b)$$

In Eq. (3.11-4), we have evaluated the derivatives at the fixed point (X_{1o}, X_{2o}), and the ellipsis indicates all derivatives higher than the first, which we are ignoring. (Note that we use partial derivatives in the Taylor series expansion because the functions depend on both X_1 and X_2.) It is useful to introduce new variables $x_1 = (X_1 - X_{1o})$ and $x_2 = (X_2 - X_{2o})$, which indicate the deviation away from the fixed point.

$$\dot{X}_1 = f_1(X_1, X_2) = (X_1 - X_{1o})\frac{\partial f_1}{\partial X_1} + (X_2 - X_{2o})\frac{\partial f_1}{\partial X_2} + \ldots \qquad (3.11\text{-}4a)$$

$$\dot{X}_2 = f_2(X_1, X_2) = (X_1 - X_{1o})\frac{\partial f_2}{\partial X_1} + (X_2 - X_{2o})\frac{\partial f_2}{\partial X_2} + \ldots \qquad (3.11\text{-}4b)$$

$$x_1 = (X_1 - X_{1o}) \text{ and } x_2 = (X_2 - X_{2o}).$$

Noting that

$$\dot{x}_1 = \dot{X}_1 \text{ and } \dot{x}_2 = \dot{X}_2 \qquad (3.11\text{-}5)$$

and ignoring all the higher-order derivative terms, we may write Eq. (3.11-4) as

$$\begin{aligned} \dot{X}_1 &= f_1(X_1.X_2) \\ \dot{X}_2 &= f_2(X_1.X_2) \end{aligned} \longrightarrow \left\{ \begin{aligned} \dot{x}_1 &= \frac{\partial f_1}{\partial x_1}x_1 + \frac{\partial f_1}{\partial x_2}x_2 \\ \dot{x}_2 &= \frac{\partial f_2}{\partial x_1}x_1 + \frac{\partial f_2}{\partial x_2}x_2 \end{aligned} \right. \qquad (3.11\text{-}6)$$

Equazioni linearizzate

Please note that Eqs. (3.11-6) are *linear*, first-order differential equations with constant coefficients (the factors multiplying x_1 and x_2 are independent of time) for the new state variables x_1 and x_2. There are many standard techniques for solving such differential equations. We shall use a method that gets us to the desired results as quickly as possible.

$$\dot{x}_1 = \frac{\partial f_1}{\partial x_1} x_1 + \frac{\partial f_1}{\partial x_2} x_2$$
$$\dot{x}_2 = \frac{\partial f_2}{\partial x_1} x_1 + \frac{\partial f_2}{\partial x_2} x_2$$

To simplify the notation, we shall write

$$f_{ij} = \frac{\partial f_i}{\partial x_j} \tag{3.11-7}$$

where i and j = 1 or 2. First, we find a differential equation for x_1 alone by differentiating the first equation in Eq. (3.11-6) with respect to time and then eliminating $\dot{x}_2$ by the use of the second equation in Eq. (3.11-6):

$$\begin{aligned} \ddot{x}_1 &= f_{11}\dot{x}_1 + f_{12}\dot{x}_2 \\ &= f_{11}\dot{x}_1 + f_{12}(f_{21}x_1 + f_{22}x_2) \end{aligned} \tag{3.11-8}$$

We now use the first of Eq. (3.11-6) again to eliminate x_2:

$$\ddot{x}_1 = (f_{11} + f_{22})\dot{x}_1 + (f_{12}f_{21} - f_{11}f_{22})x_1 \tag{3.11-9}$$

To solve Eq. (3.11-9), let us assume that the solution can be written in the form

soluzione particolare

$$x_1(t) = Ce^{\lambda t} \tag{3.11-10}$$

where λ is a constant to be determined, and C is a constant (independent of time) to be determined from the initial ($t = 0$) conditions. Let us pause a second to note that if λ is positive (and real) then the trajectory will be repelled by the fixed point; that is, we have an unstable fixed point. If λ is negative (and real), then the trajectory approaches the fixed point; that is, we have a stable fixed point. As we shall see later, λ may also be a complex number.

$$\ddot{x}_1 = (f_{11} + f_{22})\dot{x}_1 + (f_{12}f_{21} - f_{11}f_{22})x_1 \tag{3.11-9}$$

$$x_1(t) = Ce^{\lambda t} \tag{3.11-10}$$

Let us return to our solution. If we use Eq. (3.11-10) in Eq. (3.11-9), then we find that

$$\lambda^2 - (f_{11} + f_{22})\lambda + (f_{11}f_{22} - f_{12}f_{21}) = 0 \tag{3.11-11}$$

We call Eq. (3.11-11) the ***characteristic equation*** for λ, whose value depends only on the derivatives of the time evolution functions evaluated at the fixed point. Eq. (3.11-11) is a quadratic equation for λ and in general has two solutions, which we can write down from the standard quadratic formula:

$$\lambda_{\pm} = \frac{f_{11} + f_{22} \pm \sqrt{(f_{11} + f_{22})^2 - 4(f_{11}f_{22} - f_{12}f_{21})}}{2} \tag{3.11-12}$$

We have denoted λ_+ as the result obtained with the + sign in front of the square root in Eq. (3.11-12) and λ_- the result obtained with the – sign. Obviously, the characteristic values will be real numbers if the argument under the square root sign in Eq. (3.11-12) is positive. They will be complex numbers if the argument is negative.

The most general solution of Eq. (3.11-9) can then be written as

soluzione generale

$$x_1(t) = Ce^{(\lambda_+)t} + De^{(\lambda_-)t} \tag{3.11-13}$$

where C and D are constants that can be found from the initial conditions $x_1(t=0)$ and $x_2(t=0)$.

$$\lambda_{\pm} = \frac{f_{11} + f_{22} \pm \sqrt{(f_{11} + f_{22})^2 - 4(f_{11}f_{22} - f_{12}f_{21})}}{2} \qquad (3.11\text{-}12)$$

3.12 Dynamics and Complex Characteristic Values

What are the dynamics of the system when the characteristic values are not real, but are complex numbers? This situation occurs when the argument of the square root in Eq. (3.11-12) for the characteristic values is negative. We shall find that this case describes behavior in which trajectories spiral in toward or away from the fixed point, as illustrated in Fig. 3.10.

When the argument of the square root in Eq. (3.11-12) is negative, we may write the characteristic values as

$$\lambda_{\pm} = R \pm i\Omega \qquad (3.12\text{-}1)$$

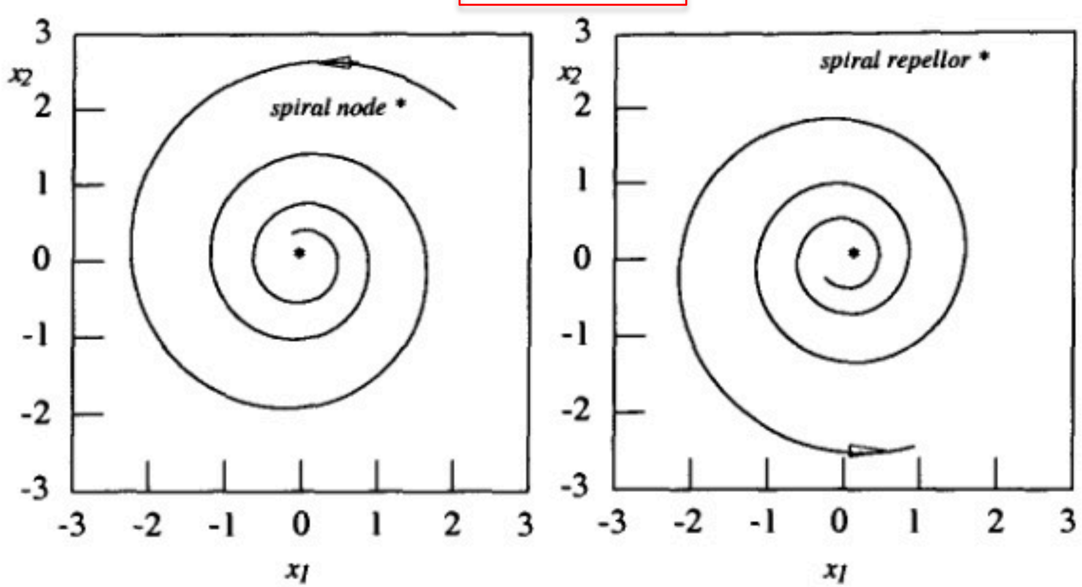


Fig. 3.10. A spiral node (left) and a spiral repellor (right) occur when the characteristic values of a fixed point are complex numbers.

$$\boxed{\lambda_{\pm} = R \pm i\Omega}$$ where $$i = \sqrt{-1}$$

$$\boxed{\lambda_{\pm} = \frac{f_{11} + f_{22} \pm \sqrt{(f_{11} + f_{22})^2 - 4(f_{11}f_{22} - f_{12}f_{21})}}{2}}$$

$$R = \frac{1}{2}(f_{11} + f_{22})$$

$$\Omega = \frac{1}{2}\sqrt{\left|(f_{11} + f_{22})^2 - 4(f_{11}f_{22} - f_{12}f_{21})\right|} \qquad (3.12\text{-}2)$$

Using the standard mathematical language of complex numbers, we say that R is the "real part" and Ω is the "imaginary part" of these complex numbers. The two eigenvalues λ_+ and λ_- form a complex conjugate pair: λ_+ is the complex conjugate of λ_- and vice versa. To see what the trajectory behavior is like in this case, we use these characteristic values in the equation for $x_1(t)$:

$$\begin{aligned} x_1(t) &= Ce^{(\lambda_+)t} + De^{(\lambda_-)t} \\ &= e^{Rt}\left[Ce^{i\Omega t} + De^{-i\Omega t}\right] \end{aligned} \qquad (3.12\text{-}3)$$

To see what is going on, let us consider the special case $x_1(0) = 0$, which tells us that $C = -D$. We now use the famous Euler formula

$$e^{i\theta} = \cos\theta + i\sin\theta \longrightarrow \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

to write

$$x_1(t) = Fe^{Rt}\sin(\Omega t) \qquad (3.12\text{-}5)$$

$$x_1(t) = Fe^{Rt}\sin(\Omega t) \longrightarrow F = \frac{f_{12}x_2(0)}{\Omega}$$

where F is a constant that depends on $x_2(0)$. From this result we see that x_1 oscillates in time with an angular frequency Ω while the amplitude of the oscillation increases or decreases exponentially (depending on whether $R > 0$ or $R < 0$). x_2 undergoes similar behavior. The corresponding state space behavior is shown schematically (with different initial conditions) in Fig. 3.10. For more general initial conditions, the state space behavior is still the same: oscillations with exponentially increasing or decreasing amplitude.

$$\begin{cases} x_1(t) = F_1 e^{Rt}\sin\Omega t \\ x_2(t) = F_2 e^{Rt}\sin\Omega t \end{cases}$$

$$F_1 = \frac{f_{12}x_2(0)}{\Omega}$$

$$F_2 = \frac{f_{21}x_1(0)}{\Omega}$$

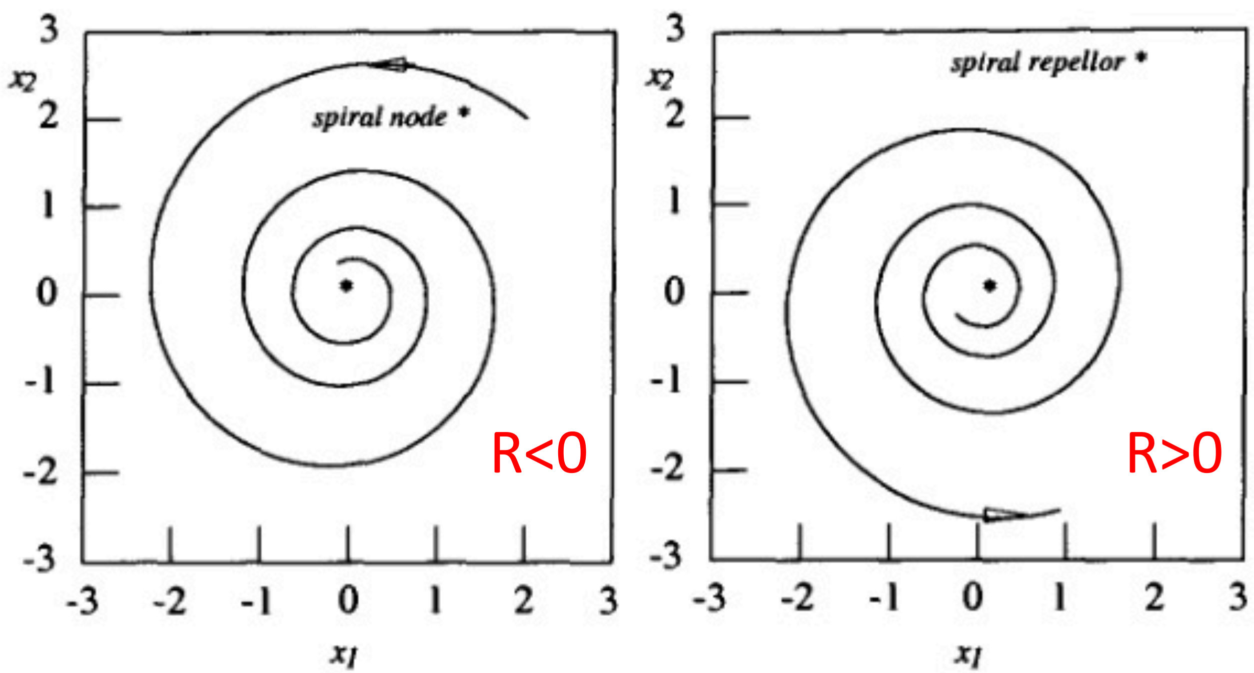


Fig. 3.10. A spiral node (left) and a spiral repellor (right) occur when the characteristic values of a fixed point are complex numbers.

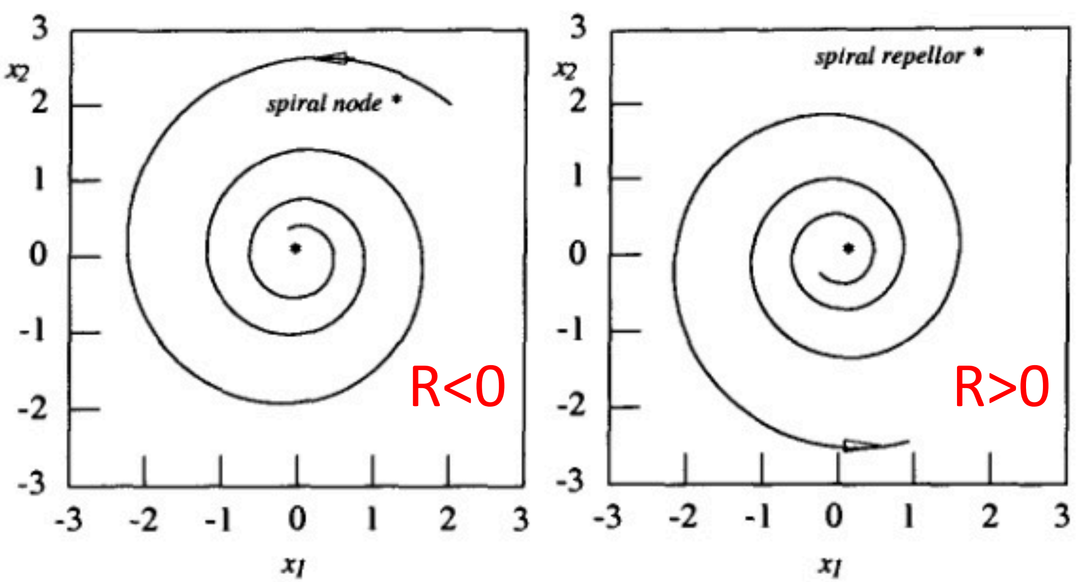
For the fixed point on the left in Fig. 3.10, we say we have a ***spiral node*** (sometimes called a ***focus***) since the trajectories spiral in toward the fixed point. On the right in Fig. 3.10, we have a ***spiral repellor*** (sometimes called an ***unstable focus***). In the special case when $R = 0$, the trajectory forms a closed loop around the fixed point. This closed loop trajectory is called a ***cycle***. If trajectories in the neighborhood of this cycle are attracted toward it as time goes on, then the cycle is called a ***limit cycle***. We need a more detailed analysis to see if this cycle is itself stable or unstable. An analysis of cycle behavior will be taken up in Section 3.16.

It is important to realize that the spiral type behavior shown in Fig. 3.10 and the cycle type behavior discussed in the Section 3.16 are possible only in state spaces of two (or higher) dimensions. They cannot occur in a one-dimensional state space because of the No-Intersection Theorem (recall Exercise 3.8-2).

$$\begin{cases} x_1(t) = F_1 e^{Rt} \sin \Omega t \\ x_2(t) = F_2 e^{Rt} \sin \Omega t \end{cases}$$

$$R = \frac{1}{2}(f_{11} + f_{22})$$

$$\Omega = \frac{1}{2}\sqrt{\left|(f_{11}+f_{22})^2 - 4(f_{11}f_{22} - f_{12}f_{21})\right|}$$



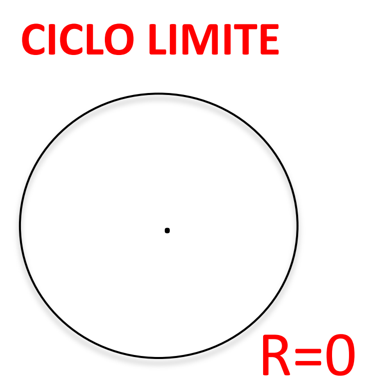


Fig. 3.10. A spiral node (left) and a spiral repellor (right) occur when the characteristic values of a fixed point are complex numbers.