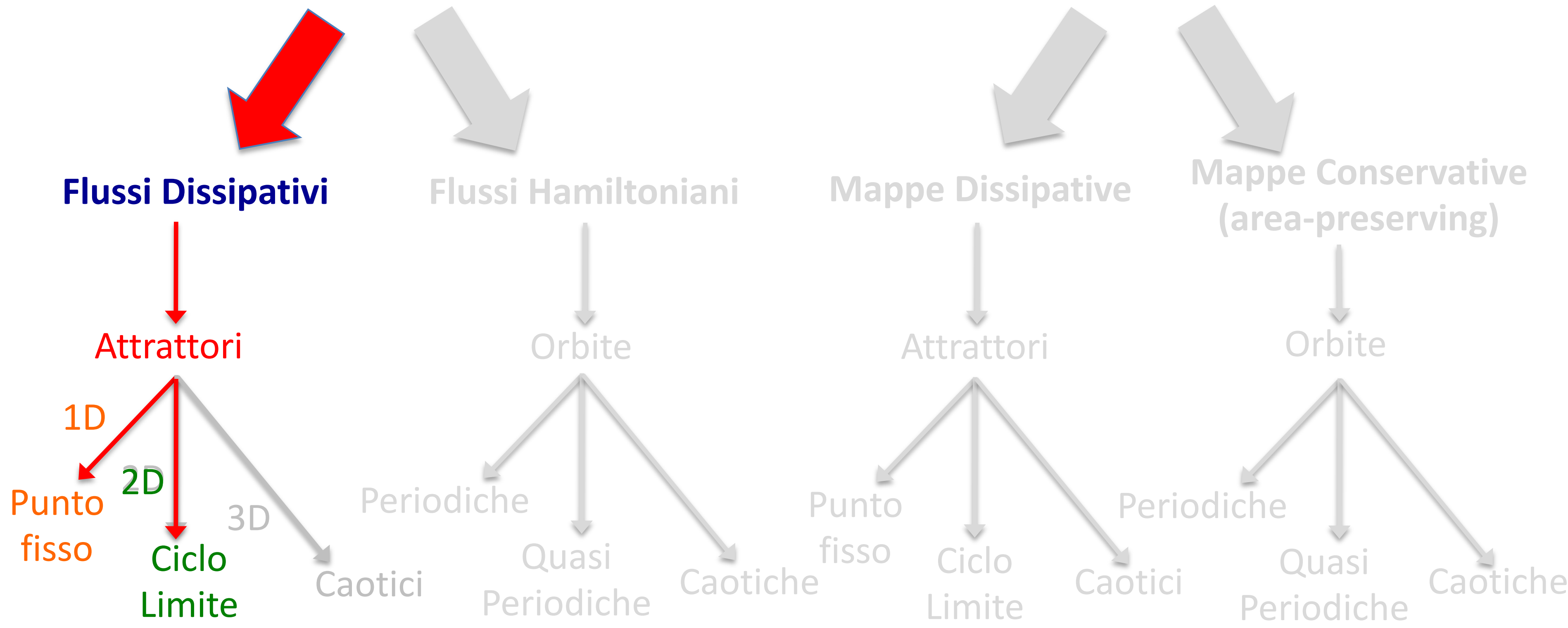
Classificazione dei Sistemi Dinamici

Sistemi dinamici continui (Flussi)

Sistemi dinamici discreti (Mappe)

$$\dot{X} = f(X)$$

$$x_{n+1} = Ax_n(1-x_n) \equiv f_A(x)$$

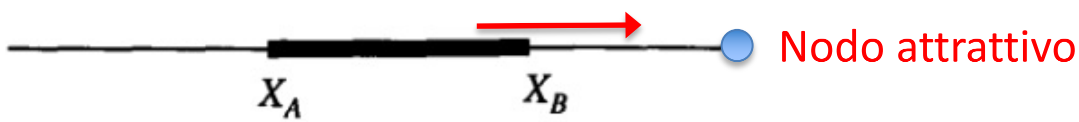


$$\dot{X} = f(X)$$

Flussi dissipativi in una dimensione

$$\frac{1}{L}\frac{dL}{dt} = \frac{1}{L}[f(X_B) - f(X_A)] = \frac{df(X)}{dX} < 0$$

fixed points (dim.0)



A "cluster of initial conditions," indicated by the heavy line, along the X axis.

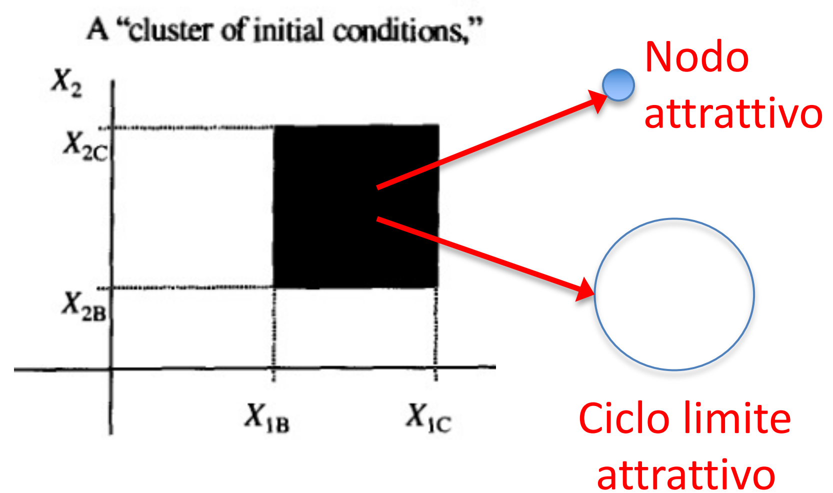
$$\dot{X}_1 = f_1(X_1, X_2)$$
$$\dot{X}_2 = f_2(X_1, X_2)$$

Flussi dissipativi in due dimensioni

$$\frac{1}{A}\frac{dA}{dt} = \frac{\partial f_1}{\partial X_1} + \frac{\partial f_2}{\partial X_2} < 0$$

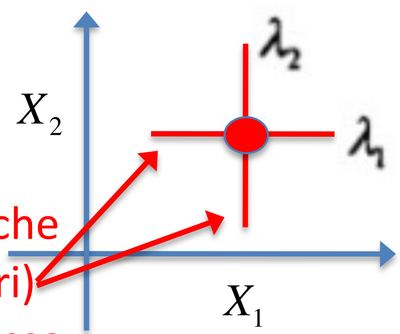
fixed points (dim.0)

limit cycles (dim.1)



A Special Case

Before considering the general problem of fixed point characteristics in two dimensions, let us first look at a particularly simple case—the case for which only two of the four derivatives are not equal to 0. In particular, let us assume that at the fixed point (X_{1o}, X_{2o}) the derivatives have the following values:



Direzioni Caratteristiche (autovettori) del Punto Fisso

$$\frac{\partial f_1}{\partial X_1} = \lambda_1 \qquad \frac{\partial f_1}{\partial X_2} = 0$$

$$\frac{\partial f_2}{\partial X_1} = 0 \qquad \frac{\partial f_2}{\partial X_2} = \lambda_2 \qquad (3.10\text{-}4)$$

Valori Caratteristici (autovalori) del Punto Fisso

In this special case, what happens along the X_1 direction in the neighborhood of the fixed point depends only on λ_1, and what happens along the X_2 direction depends only on λ_2. For this case, we say that the X_1 and X_2 axes are the ***characteristic directions*** with the associated characteristic values λ_1 and λ_2. (Please keep in mind that this independence of the X_1 and X_2 motions holds only in this special case and only in the vicinity of this fixed point.)

Types of Fixed Points in Two Dimensions

We can now begin to construct the catalog of types of fixed points in two dimensions by fitting together the possible types of one-dimensional behavior. We shall soon see, however, that there are new types of behavior possible in two dimensions. In the simplest case, λ_1 and λ_2 are both real numbers and both are nonzero. (When a characteristic value equals 0, then we need a more complicated analysis, just as we did in one-dimension.) By using arguments like those leading up to Eq. (3.6-3), we can see that there are four possible fixed points as listed in Table 3.2. In Fig. 3.8, sample trajectories are shown in the neighborhood of those fixed points.

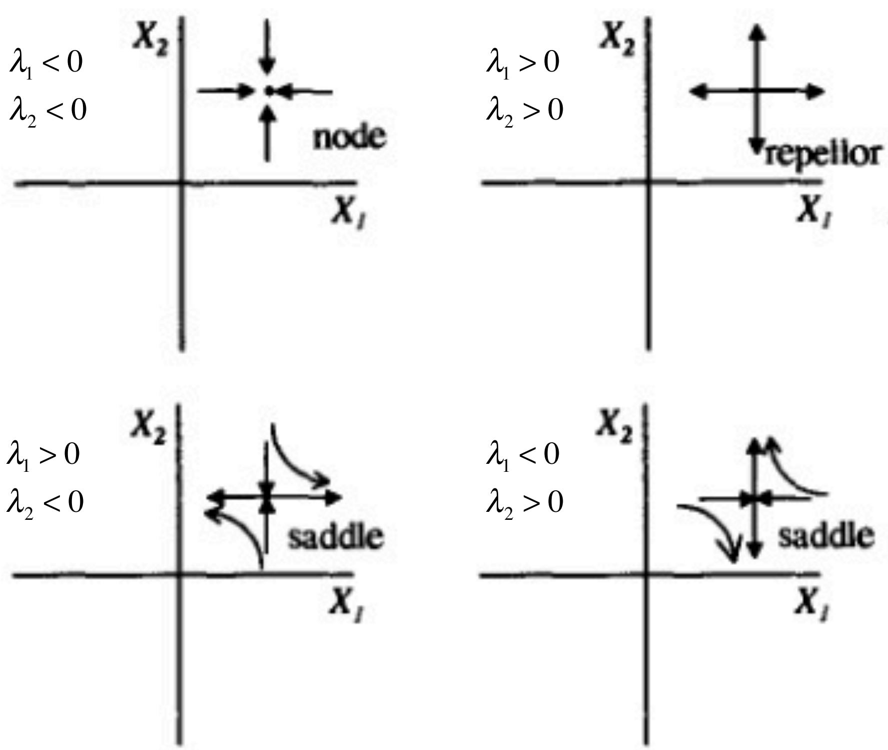


Fig. 3.8. Sample trajectories near each of the four types of fixed points with real characteristic values in two dimensions.

Table 3.2.
Possible Fixed Point Characters
with Real Characteristic Values

λ_1	λ_2	Type of Fixed Point
< 0	< 0	attracting node
> 0	> 0	repellor
> 0	< 0	saddle point
< 0	> 0	saddle point

Punti Fissi in due dimensioni: il caso generale

$$\dot{X}_1 = f_1(X_1, X_2) = (X_1 - X_{1o})\frac{\partial f_1}{\partial X_1} + (X_2 - X_{2o})\frac{\partial f_1}{\partial X_2} + \ldots \qquad (3.11\text{-}4a)$$

$$\dot{X}_2 = f_2(X_1, X_2) = (X_1 - X_{1o})\frac{\partial f_2}{\partial X_1} + (X_2 - X_{2o})\frac{\partial f_2}{\partial X_2} + \ldots \qquad (3.11\text{-}4b)$$

$x_1 = (X_1 - X_{1o})$ and $x_2 = (X_2 - X_{2o})$.

Noting that

$$\dot{x}_1 = \dot{X}_1 \text{ and } \dot{x}_2 = \dot{X}_2 \qquad (3.11\text{-}5)$$

and ignoring all the higher-order derivative terms, we may write Eq. (3.11-4) as

$$\begin{aligned}\dot{X}_1 &= f_1(X_1.X_2)\\ \dot{X}_2 &= f_2(X_1.X_2)\end{aligned} \longrightarrow \begin{cases}\dot{x}_1 = \dfrac{\partial f_1}{\partial x_1}x_1 + \dfrac{\partial f_1}{\partial x_2}x_2\\[2ex] \dot{x}_2 = \dfrac{\partial f_2}{\partial x_1}x_1 + \dfrac{\partial f_2}{\partial x_2}x_2\end{cases} \qquad (3.11\text{-}6)$$

Equazioni linearizzate

Please note that Eqs. (3.11-6) are *linear*, first-order differential equations with constant coefficients (the factors multiplying x_1 and x_2 are independent of time) for the new state variables x_1 and x_2. There are many standard techniques for solving such differential equations. We shall use a method that gets us to the desired results as quickly as possible.

$$\lambda_{\pm} = \frac{f_{11} + f_{22} \pm \sqrt{(f_{11} + f_{22})^2 - 4(f_{11}f_{22} - f_{12}f_{21})}}{2} \qquad (3.11\text{-}12)$$

3.12 Dynamics and Complex Characteristic Values

What are the dynamics of the system when the characteristic values are not real, but are complex numbers? This situation occurs when the argument of the square root in Eq. (3.11-12) for the characteristic values is negative. We shall find that this case describes behavior in which trajectories spiral in toward or away from the fixed point, as illustrated in Fig. 3.10.

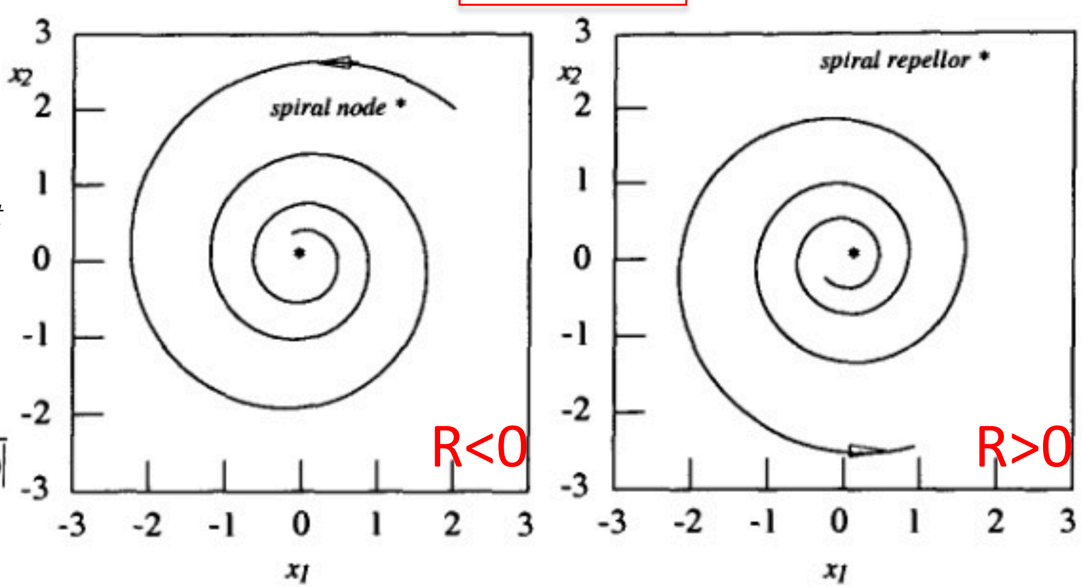
When the argument of the square root in Eq. (3.11-12) is negative, we may write the characteristic values as

$$\lambda_{\pm} = R \pm i\Omega \qquad (3.12\text{-}1)$$

$$\begin{cases} x_1(t) = F_1 e^{Rt} \sin \Omega t \\ x_2(t) = F_2 e^{Rt} \sin \Omega t \end{cases}$$

$$R = \frac{1}{2}(f_{11} + f_{22})$$

$$\Omega = \frac{1}{2}\sqrt{\left|(f_{11} + f_{22})^2 - 4(f_{11}f_{22} - f_{12}f_{21})\right|}$$



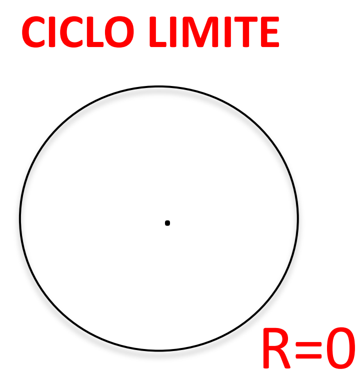


Fig. 3.10. A spiral node (left) and a spiral repellor (right) occur when the characteristic values of a fixed point are complex numbers.

Metodo dello Jacobiano per studiare i punti fissi nel caso generale a 2 dim.

Equazioni originarie

$$\dot{X}_1 = f_1(X_1, X_2)$$
$$\dot{X}_2 = f_2(X_1, X_2)$$

…ricavare i punti fissi…

Equazioni linearizzate nelle vicinanze di un dato punto fisso (X_{1o}, X_{2o})

$$\dot{x}_1 = \frac{\partial f_1}{\partial x_1} x_1 + \frac{\partial f_1}{\partial x_2} x_2$$
$$\dot{x}_2 = \frac{\partial f_2}{\partial x_1} x_1 + \frac{\partial f_2}{\partial x_2} x_2$$

with $f_{ij} = \frac{\partial f_i}{\partial x_j}$

…calcolate nel punto fisso

Distanze dal punto fisso

$$x_1 = X_1 - X_{1o}$$
$$x_2 = X_2 - X_{2o}$$

3.14 The Jacobian Matrix for Characteristic Values

We would now like to introduce a more elegant and general method of finding the characteristic equation for a fixed point. This method makes use of the so-called ***Jacobian matrix*** of the derivatives of the time evolution functions. Once we see how this procedure works, it will be easy to generalize the method, at least in principle, to find characteristic values for fixed points in state spaces of any dimension. The Jacobian matrix for the system is defined to be the following square array of the derivatives:

Matrice Jacobiana

$$J = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \tag{3.14-1}$$

Autovalori: λ_+, λ_-

where the derivatives are evaluated at the fixed point. We subtract λ from each of the principal diagonal (upper left to lower right) elements and set the determinant of the matrix equal to 0:

Eq. agli autovalori

$$J = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

$$\boxed{J\vec{v} = \lambda\vec{v}} \rightarrow \det(J - \lambda I) = 0 \rightarrow \begin{vmatrix} f_{11} - \lambda & f_{12} \\ f_{21} & f_{22} - \lambda \end{vmatrix} = 0$$

Equazione caratteristica dello Jacobiano

$$\lambda^2 - (f_{11} + f_{22})\lambda + (f_{11}f_{22} - f_{12}f_{21}) = 0 \qquad (3.11\text{-}11)$$

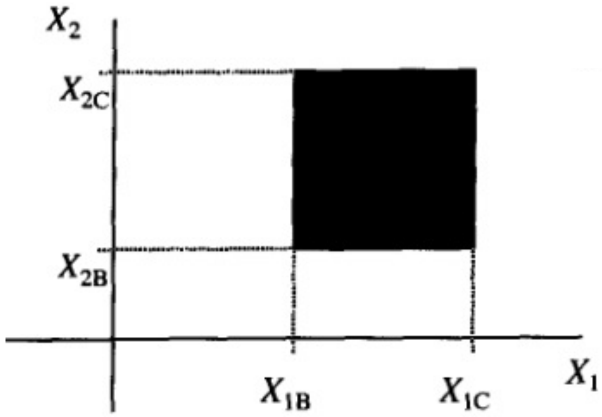
$$\lambda_{\pm} = \frac{f_{11} + f_{22} \pm \sqrt{(f_{11} + f_{22})^2 - 4(f_{11}f_{22} - f_{12}f_{21})}}{2} \qquad (3.11\text{-}12)$$

Multiplying out the determinant in the usual way then yields the characteristic equation (3.11-11). The Jacobian matrix method is obviously easily extended to *d*-dimensions by writing down the *d*-by-*d* matrix of derivatives of the *d* time-evolution functions f_n, forming the corresponding determinant, and then (at least in principle) solving the resulting *d*th order equation for the characteristic values.

We now introduce some terminology from linear algebra to make some very general and very powerful statements about the characteristic values for a given fixed point. First, the ***trace*** of a matrix, such as the Jacobian matrix (3.14-1), is defined to be the sum of the principal diagonal elements. For Eq. (3.14-1) this is explicitly

Traccia dello Jacobiano

$$TrJ = f_{11} + f_{22} \qquad (3.14\text{-}3)$$



Condizione affinchè un cluster di condizioni iniziali collassi su un attrattore stabile:

$$\frac{1}{A}\frac{dA}{dt} = (f_{11} + f_{22}) < 0$$

$$TrJ < 0$$

$$x_1(t) = F_1 e^{Rt} \sin \Omega t$$
$$x_2(t) = F_2 e^{Rt} \sin \Omega t$$

$$R = \frac{1}{2}(f_{11} + f_{22})$$

According to Eq. (3.13-5), however, this is just the combination of derivatives needed to test whether or not the system's trajectories collapse toward an attractor. To make a connection with the previous section, we note that $TrJ = 2R$, so that we see that the sign of TrJ determines whether the fixed point is a node or a repellor.

Linear algebra also tells us how to find the directions to be associated with the characteristic values. For a saddle point, these directions will be the directions for the in-sets and out-sets in the immediate neighborhood of the saddle point. The basic idea is that by transforming the coordinate system, (in general the new coordinates are linear combinations of the original coordinates), we can bring the Jacobian matrix to the so-called diagonal form in which only the principal diagonal elements are non-zero. In that case the matrix has the form (for a two-dimensional state space)

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad (3.14\text{-}5)$$

In linear algebra this procedure is called "finding the eigenvalues and eigenvectors of the matrix." For our purposes, the eigenvalues are the characteristic values of the fixed point and the eigenvectors give the associated characteristic directions. However, we will not need these eigenvectors for most of our purposes. The interested reader is referred to the books on linear algebra listed at the end of the chapter.

We now introduce one more symbol:

$$J = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

Determinante dello Jacobiano

$$\Delta = f_{11}f_{22} - f_{21}f_{12} \qquad (3.14\text{-}6)$$

Δ is called the ***determinant*** of that matrix. Then we may show that the nature of the fixed point is determined by TrJ and Δ as listed in Table 3.3.

$$\lambda_{\pm} = \frac{f_{11} + f_{22} \pm \sqrt{(f_{11} + f_{22})^2 - 4(f_{11}f_{22} - f_{12}f_{21})}}{2} \rightarrow \lambda_{\pm} = \frac{TrJ \pm \sqrt{(TrJ)^2 - 4\Delta}}{2}$$

$$\lambda_{\pm} = R \pm i\Omega$$

$$x_1(t) = F_1 e^{Rt} \sin\Omega t$$

$$x_2(t) = F_2 e^{Rt} \sin\Omega t$$

$$\begin{cases} R = \frac{1}{2} TrJ \\ \Omega = \frac{1}{2}\sqrt{TrJ^2 - 4\Delta} \end{cases}$$

Table 3.3

Fixed Points for Two-dimensional State Space

	$TrJ < 0$	$TrJ > 0$
$\Delta > (1/4)(TrJ)^2$	spiral node	spiral repellor
$0 < \Delta < (1/4)(TrJ)^2$	node	repellor
$\Delta < 0$	saddle point	saddle point

Riepilogo dei Punti Fissi in uno Spazio degli Stati a Due Dimensioni

$$\lambda_{\pm} = \frac{TrJ \pm \sqrt{(TrJ)^2 - 4\Delta}}{2}$$

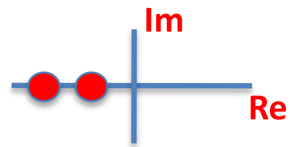
con:
$$\begin{cases} TrJ = f_{11} + f_{22} \\ \Delta = f_{11}f_{22} - f_{21}f_{12} \end{cases}$$

$\Delta > 0$

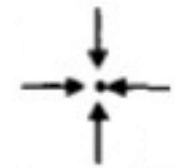
- $(TrJ)^2 - 4\Delta > 0$

  $0 < \Delta < \frac{1}{4}(TrJ)^2 \rightarrow \lambda_+, \lambda_-$ **reali e concordi**

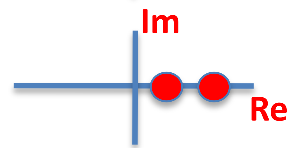
  - $TrJ < 0$

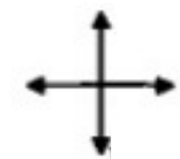
    NODE

  - $TrJ > 0$

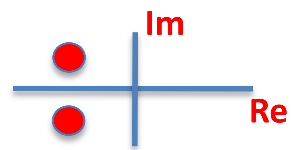
    REPELLOR

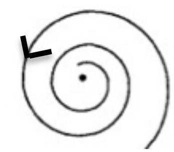
- $(TrJ)^2 - 4\Delta < 0$

  $\Delta > \frac{1}{4}(TrJ)^2 \rightarrow \lambda_+, \lambda_-$ **complessi coniugati**

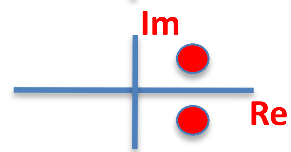
  - $TrJ < 0$

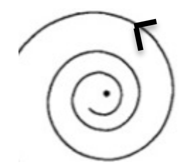
    SPIRAL NODE

  - $TrJ > 0$

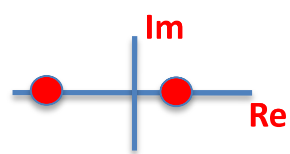
    

    SPIRAL REPELLOR

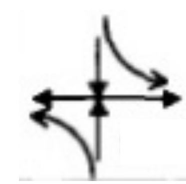
    

$\Delta < 0 \rightarrow \lambda_+, \lambda_-$ **reali e discordi**

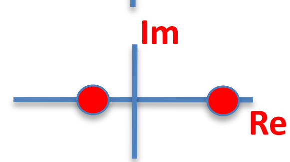
- $TrJ < 0$

  SADDLE POINT

- $TrJ > 0$

  SADDLE POINT

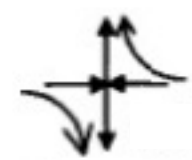
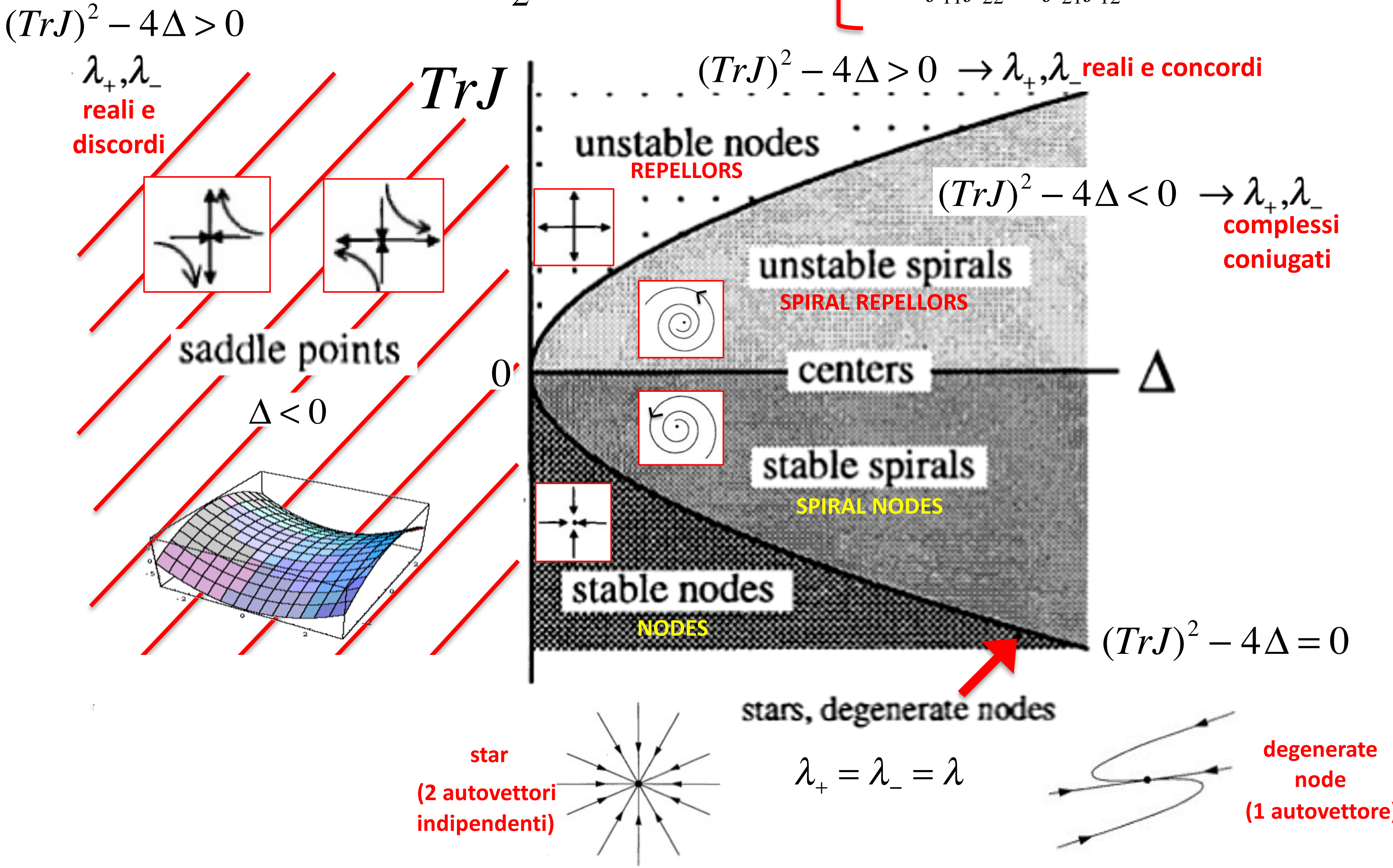
  

Diagramma dei Punti Fissi in uno Spazio degli Stati a Due Dimensioni

$$\lambda_{\pm} = \frac{TrJ \pm \sqrt{(TrJ)^2 - 4\Delta}}{2}$$

con:
$$\begin{cases} TrJ = f_{11} + f_{22} \\ \Delta = f_{11}f_{22} - f_{21}f_{12} \end{cases}$$



Summary of Fixed Point Analysis for Two-dimensional State Space

1. Write the time evolution equations in the first-order time derivative form of Eq. (3.10-1).

$$\dot{X}_1 = f_1(X_1, X_2)$$
$$\dot{X}_2 = f_2(X_1, X_2) \qquad (3.10\text{-}1)$$

2. Find the fixed points of the evolution by finding those points that satisfy

$$f_1(X_1, X_2) = 0$$
$$f_2(X_1, X_2) = 0$$

3. At the fixed points, evaluate the partial derivatives of the time evolution functions to set up the Jacobian matrix

$$J \equiv \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \qquad (3.14\text{-}1)$$

4. Evaluate the trace and determinant of the Jacobian matrix at the fixed point and use Table 3.3 to find the type of fixed point.
5. Use Eq. (3.11-12) to find the numerical values of the characteristic values and to specify the behavior of the state-space trajectories near the fixed point with Eq. (3.11-13).

Examples by Topic

Matrices & Linear Algebra

Matrix Arithmetic

do basic arithmetic on matrices

{{0,-1},{1,0}}.{{1,2},{3,4}}+{{2,-1},{-1,2}}

{{2,-1,1},{0,-2,1},{1,-2,0}}.{x,y,z}

Matrix Operations

compute properties of a matrix

{{6, -7, 10}, {0, 3, -1}, {0, 5, -7}}

compute the rank of a matrix

rank {{6, -11, 13}, {4, -1, 3}, {3, 4, -2}}

compute the inverse of a matrix

inv {{10, -9, -12}, {7, -12, 11}, {-10, 10, 3}}

inverse {{a, b}, {c, d}}

{{2,3},{4,5}}^(-1)

compute the adjugate of a matrix

adjugate {{8,7,7},{6,9,2},{-6,9,-2}}

Trace

compute the trace of a matrix

tr {{9, -6, 7}, {-9, 4, 0}, {-8, -6, 4}}

Determinant

compute the determinant of a matrix

determinant of {{3,4},{2,1}}

det({{9, 3, 5}, {-6, -9, 7}, {-1, -8, 1}})

det {{a, b, c}, {d, e, f}, {g, h, j}}

Row Reduction

row reduce a matrix

row reduce {{2,1,0,-3},{3,-1,0,1},{1,4,-2,-5}}

row reduction calculator

Eigenvalues & Eigenvectors

compute the eigenvalues of a matrix

eigenvalues {{4,1},{2,-1}}

compute the eigenvectors of a matrix

eigenvectors {{1,0,0},{0,0,1},{0,1,0}}

compute the characteristic polynomial of a matrix

characteristic polynomial {{4,1},{2,-1}}

Diagonalization

diagonalize a matrix

diagonalize {{1,2},{3,4}}

Matrix Decompositions »

compute the LU decomposition of a square matrix

LU decomposition of {{7,3,-11},{-6,7,10},{-11,2,-2}}

compute a singular value decomposition

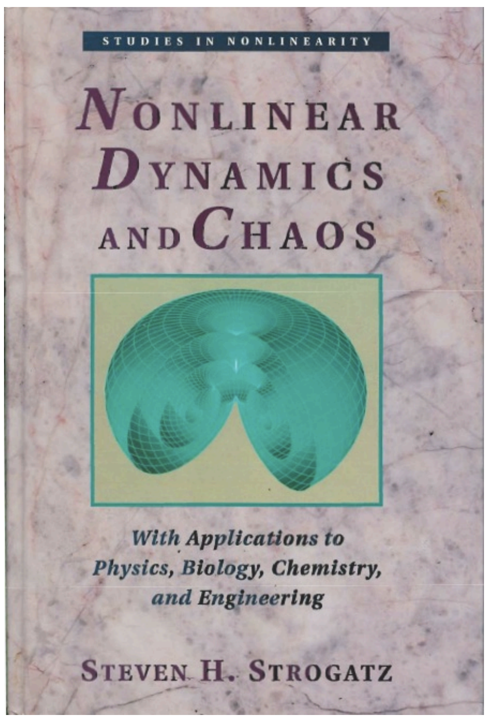
Steven
Strogatz

Sheep

Rabbit

6.4 Rabbits versus Sheep

In the next few sections we'll consider some simple examples of phase plane analysis. We begin with the classic ***Lotka–Volterra model of competition*** between two species, here imagined to be rabbits and sheep. Suppose that both species are competing for the same food supply (grass) and the amount available is limited. Furthermore, ignore all other complications, like predators, seasonal effects, and other sources of food. Then there are two main effects we should consider:



Rabbit

Steven Strogatz

Sheep

6.4 Rabbits versus Sheep

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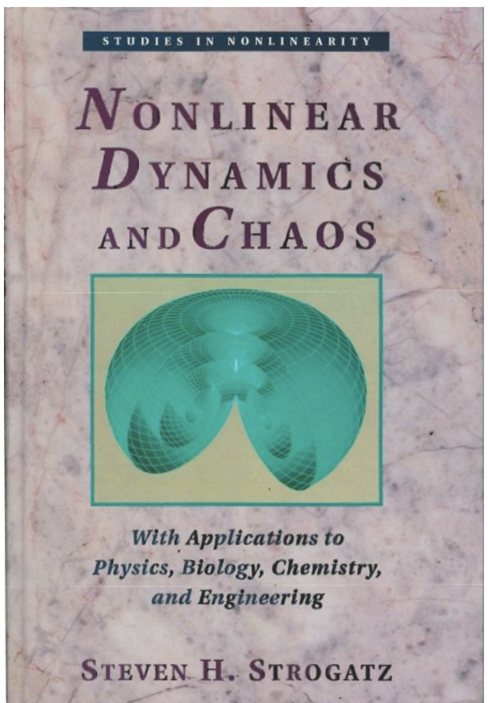
1. Each species would grow to its carrying capacity in the absence of the other. This can be modeled by assuming logistic growth for each species (recall Section 2.3). Rabbits have a legendary ability to reproduce, so perhaps we should assign them a higher intrinsic growth rate.



L'Equazione Logistica (o di Verhulst)

P.F.Verhulst
(1804-1849)

Exercise 3.8-3. The logistic differential equation. The following differential equation has a "force" term that is identical to the logistic map function introduced in Chapter 1

$$\dot{X} = AX(1-X) \qquad A \in [0,1]$$

(a) Find the fixed points for this differential equation.
(b) Determine the characteristic value and type of each of the fixed points.

Modello di crescita

Avendo supposto che il numero di individui di una popolazione sia una funzione continua del tempo $N(t)$ che ammette derivata continua, si ha che l'incremento della popolazione al variare del tempo può essere rappresentato dalla derivata di $N(t)$, che in un modello elementare si può supporre direttamente proporzionale al numero di individui della popolazione stessa.

Si ha pertanto la seguente equazione differenziale:

$$\frac{d}{dt}N = rN(t) \quad \rightarrow N(t) = N_o e^{rt}$$ *Crescita Maltusiana (esponenziale)*

con r: parametro di crescita malthusiana (tasso massimo di crescita della popolazione).

Pertanto se r è una costante la popolazione cresce in maniera esponenziale con pendenza dipendente da r.

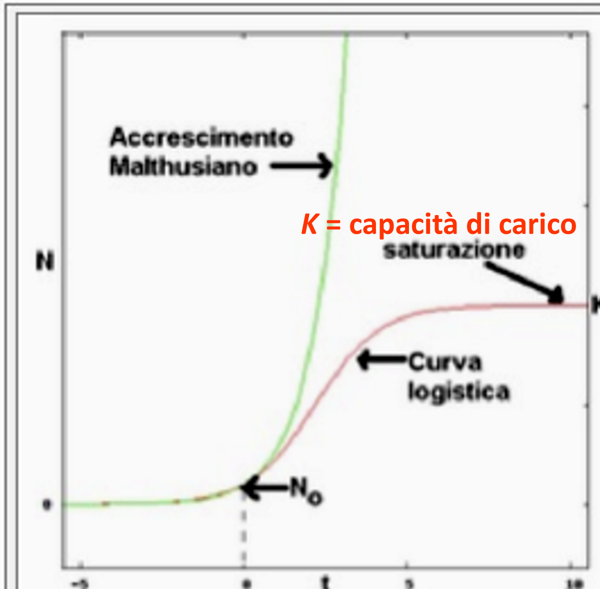
Invece in un ambiente la cui disponibilità di risorse è limitata si può descrivere l'evoluzione della popolazione utilizzando un coefficiente r che decresce all'aumentare della popolazione: il modello più semplice è $r(t) := a - bN(t)$ con a e b costanti. Sostituendo tale funzione nella precedente equazione differenziale si ottiene:

$$\frac{dN}{dt} = aN(t) - bN^2(t)$$

che può essere posta nella forma:

$$\frac{dN}{dt} = aN\left(1 - \frac{N}{K}\right) \quad \xleftrightarrow{\text{se a=b (K=1)}} \quad \dot{N}(t) = aN(1-N)$$

con $K = \frac{a}{b}$ che è la cosiddetta popolazione massima sostenibile ed a uguale al parametro di crescita malthusiana.



Confronto tra curva logistica e curva di accrescimento esponenziale (malthusiano). I parametri sono: $k = 10, N_0 = 1, r = 1$

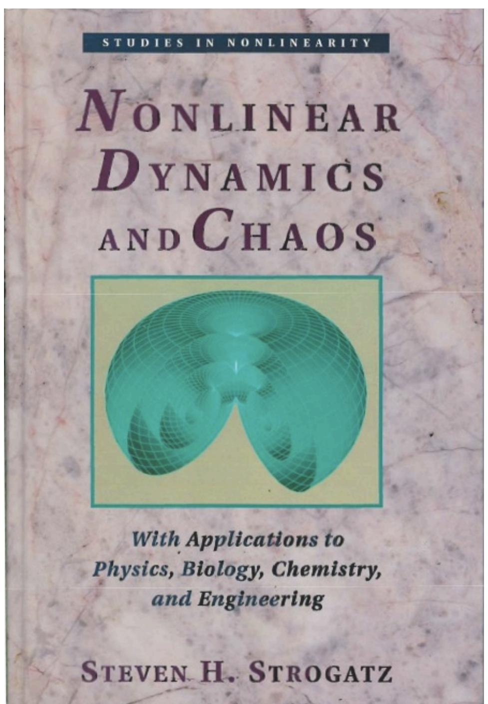
Steven Strogatz

Sheep

Rabbit

6.4 Rabbits versus Sheep

In the next few sections we'll consider some simple examples of phase plane analysis. We begin with the classic ***Lotka–Volterra model of competition*** between two species, here imagined to be rabbits and sheep. Suppose that both species are competing for the same food supply (grass) and the amount available is limited. Furthermore, ignore all other complications, like predators, seasonal effects, and other sources of food. Then there are two main effects we should consider:



1. Each species would grow to its carrying capacity in the absence of the other. This can be modeled by assuming logistic growth for each species (recall Section 2.3). Rabbits have a legendary ability to reproduce, so perhaps we should assign them a higher intrinsic growth rate.

2. When rabbits and sheep encounter each other, trouble starts. Sometimes the rabbit gets to eat, but more usually the sheep nudges the rabbit aside and starts nibbling (on the grass, that is). We'll assume that these conflicts occur at a rate proportional to the size of each population. (If there were twice as many sheep, the odds of a rabbit encountering a sheep would be twice as great.) Furthermore, we assume that the conflicts reduce the growth rate for each species, but the effect is more severe for the rabbits.

Rabbit

A specific model that incorporates these assumptions is

$$\begin{cases} \dot{x} = x(3 - x - 2y) \\ \dot{y} = y(2 - x - y) \end{cases}$$

where

$x(t)$ = population of rabbits,
$y(t)$ = population of sheep

Sheep

$$x = 0 \rightarrow \dot{x} = 0$$
$$\dot{y} = 2y(1 - \frac{y}{2})$$
$$[b = 1, a = K = 2]$$

$$y = 0 \rightarrow \dot{y} = 0$$
$$\dot{x} = 3x(1 - \frac{x}{3})$$
$$[b = 1, a = K = 3]$$

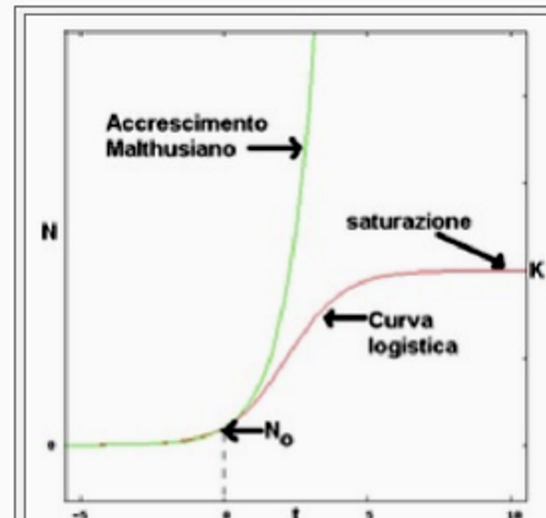
and $x, y \geq 0$. The coefficients have been chosen to reflect this scenario, but are otherwise arbitrary. In the exercises, you'll be asked to study what happens if the coefficients are changed.

L'Equazione Logistica

P.F.Verhulst (1804-1849)

$$\dot{N} = aN(1 - \frac{N}{a})$$
$$[b = 1, a = K]$$

K = capacità di carico



Confronto tra curva logistica e curva di accrescimento esponenziale (malthusiano). I parametri sono: $k = 10, N_0 = 1, r = 1$

$$x = 0 \rightarrow \dot{x} = 0$$
$$\dot{y} = 2y(1 - \frac{y}{2})$$
$$[b = 1, a = K = 2]$$

A specific model that incorporates these assumptions is

Rabbit

$$\begin{cases} \dot{x} = x(3 - x - 2y) \\ \dot{y} = y(2 - x - y) \end{cases}$$

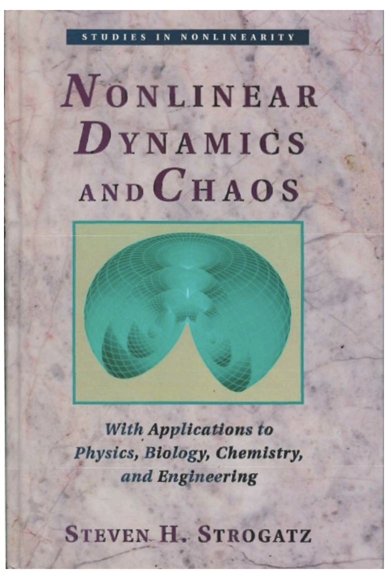
Sheep

where

$$x(t) = \text{population of rabbits,}$$
$$y(t) = \text{population of sheep}$$

$$y = 0 \rightarrow \dot{y} = 0$$
$$\dot{x} = 3x(1 - \frac{x}{3})$$
$$[b = 1, a = K = 3]$$

and $x, y \geq 0$. The coefficients have been chosen to reflect this scenario, but are otherwise arbitrary. In the exercises, you'll be asked to study what happens if the coefficients are changed.



2. When rabbits and sheep encounter each other, trouble starts. Sometimes the rabbit gets to eat, but more usually the sheep nudges the rabbit aside and starts nibbling (on the grass, that is). We'll assume that these conflicts occur at a rate proportional to the size of each population. (If there were twice as many sheep, the odds of a rabbit encountering a sheep would be twice as great.) Furthermore, we assume that the conflicts reduce the growth rate for each species, but the effect is more severe for the rabbits.

$$x=0 \rightarrow \dot{x}=0$$

$$\dot{y}=2y(1-\frac{y}{2})$$

$$[b=1, a=K=2]$$

A specific model that incorporates these assumptions is

Rabbit

$$\begin{cases} \dot{x}=x(3-x-2y) \\ \dot{y}=y(2-x-y) \end{cases}$$

Sheep

where

$x(t)=$ population of rabbits,

$y(t)=$ population of sheep

$$y=0 \rightarrow \dot{y}=0$$

$$\dot{x}=3x(1-\frac{x}{3})$$

$$[b=1, a=K=3]$$

and $x, y \geq 0$. The coefficients have been chosen to reflect this scenario, but are otherwise arbitrary. In the exercises, you'll be asked to study what happens if the coefficients are changed.

To find the fixed points for the system, we solve $\dot{x}=0$ and $\dot{y}=0$ simultaneously. Four fixed points are obtained: $(0,0)$, $(0,2)$, $(3,0)$, and $(1,1)$.

Solutions:

$x=0, \quad y=2$

$x=1, \quad y=1$

$x=3, \quad y=0$

$y=0, \quad x=0$

$$x = 0 \to \dot{x} = 0$$

$$\dot{y} = 2y(1 - \frac{y}{2})$$

$$[b = 1, a = K = 2]$$

A specific model that incorporates these assumptions is

$$\begin{cases} \dot{x} = x(3 - x - 2y) \\ \dot{y} = y(2 - x - y) \end{cases}$$

where

$$x(t) = \text{population of rabbits,}$$
$$y(t) = \text{population of sheep}$$

$$y = 0 \to \dot{y} = 0$$

$$\dot{x} = 3x(1 - \frac{x}{3})$$

$$[b = 1, a = K = 3]$$

and $x, y \geq 0$. The coefficients have been chosen to reflect this scenario, but are otherwise arbitrary. In the exercises, you'll be asked to study what happens if the coefficients are changed.

To find the fixed points for the system, we solve $\dot{x} = 0$ and $\dot{y} = 0$ simultaneously. Four fixed points are obtained: $(0,0)$, $(0,2)$, $(3,0)$, and $(1,1)$. To classify them, we compute the Jacobian:

$$A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix}.$$

Solutions:

$x = 0, \quad y = 2$

$x = 1, \quad y = 1$

$x = 3, \quad y = 0$

$y = 0, \quad x = 0$

Now consider the four fixed points in turn:

$(0,0)$: Then $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

The eigenvalues are $\lambda = 3, 2$ so $(0,0)$ is an *unstable node*. Trajectories leave the origin parallel to the eigenvector for $\lambda = 2$, i.e. tangential to $\mathbf{v} = (0,1)$, which spans the y-axis. (Recall the general rule: at a node, trajectories are tangential to the slow eigendirection, which is the eigendirection with the smallest $|\lambda|$.) Thus, the phase portrait near $(0,0)$ looks like Figure 6.4.1.

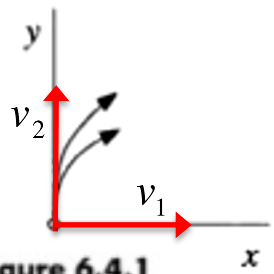
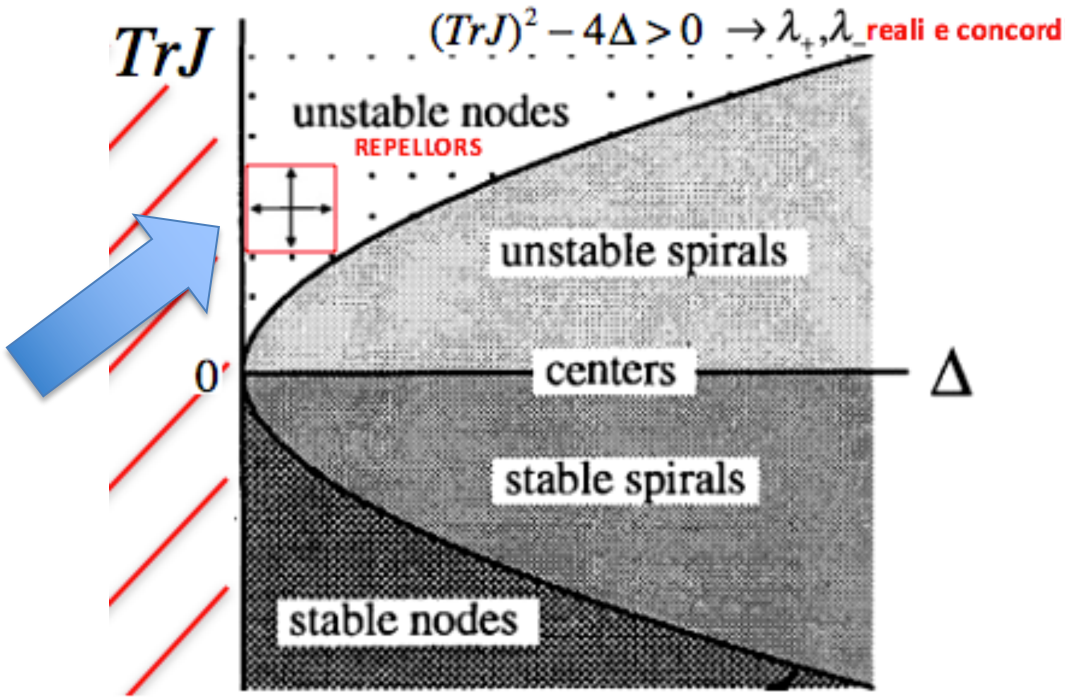


Figure 6.4.1

Eigenvalues:	Eigenvectors:
$\lambda_1 = 3$	$v_1 = (1, 0)$
$\lambda_2 = 2$	$v_2 = (0, 1)$

$$\Delta = 6 > 0$$
$$TrJ = 5 > 0$$
$$(TrJ)^2 - 4\Delta = 1 > 0$$



Eigenvalues:	Eigenvectors:
$\lambda_1 = -2$	$v_1 = (0, 1)$
$\lambda_2 = -1$	$v_2 = (-1, 2)$

$(0,2)$: Then $A = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$.

$$\Delta = 2 > 0$$
$$TrJ = -3 < 0$$
$$(TrJ)^2 - 4\Delta = 1 > 0$$

This matrix has eigenvalues $\lambda = -1, -2$, as can be seen from inspection, since the matrix is triangular. Hence the fixed point is a *stable node*. Trajectories approach along the eigendirection associated with $\lambda = -1$; you can check that this direction is spanned by $\mathbf{v} = (1, -2)$. Figure 6.4.2 shows the phase portrait near the fixed point $(0, 2)$.

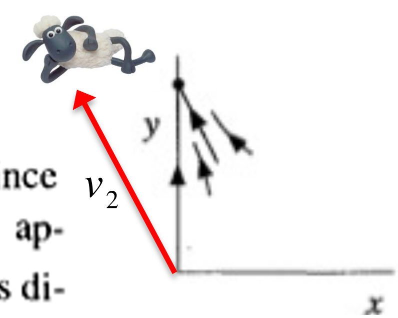
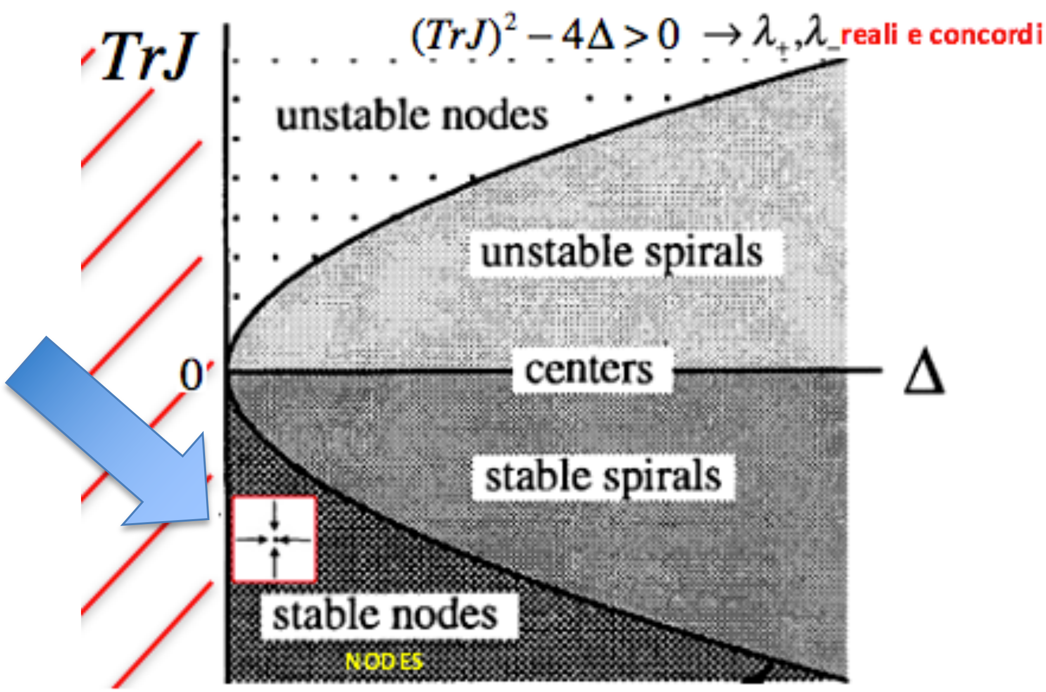


Figure 6.4.2



$(3,0)$: Then $A = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$ and $\lambda = -3, -1$.

This is also a *stable node*. The trajectories approach along the slow eigendirection spanned by $\mathbf{v} = (3,-1)$, as shown in Figure 6.4.3.

$$\Delta = 3 > 0$$
$$TrJ = -4 < 0$$
$$(TrJ)^2 - 4\Delta = 4 > 0$$

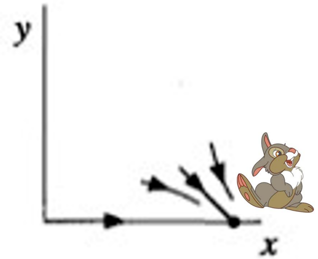


Figure 6.4.3

Eigenvalues:

$\lambda_1 = -3$

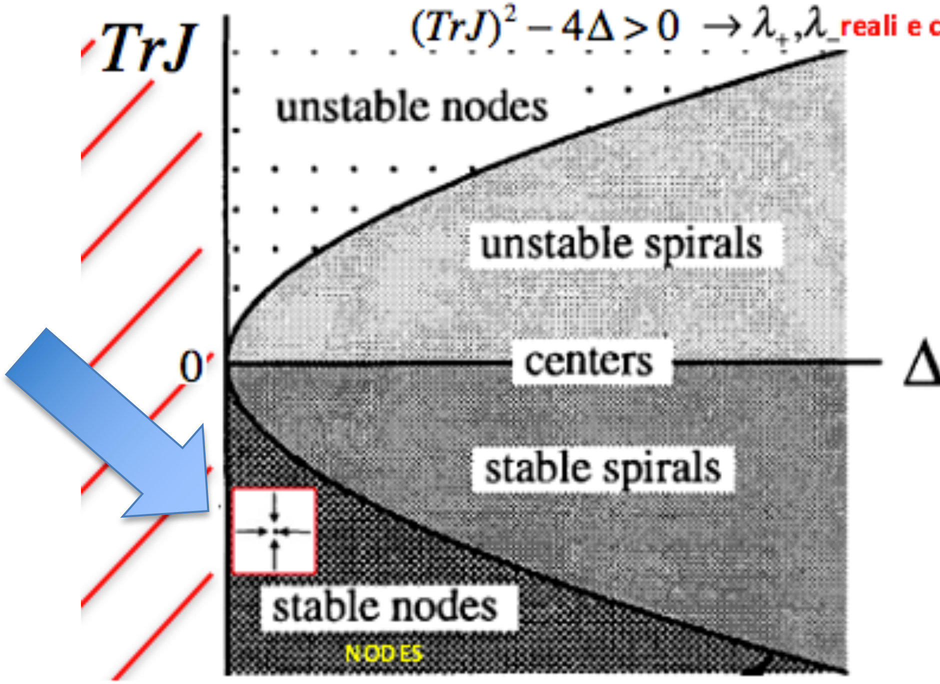
$\lambda_2 = -1$

Eigenvectors:

$v_1 = (1, 0)$

$v_2 = (-3, 1)$

$(TrJ)^2 - 4\Delta > 0 \rightarrow \lambda_+, \lambda_-$ reali e concordi



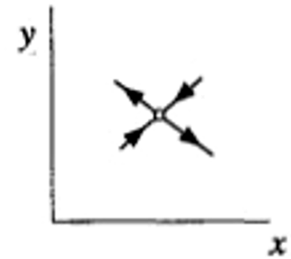
v_2

$(1,1)$: Then $A = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$, which has $\tau = -2$, $\Delta = -1$, and $\lambda = -1 \pm \sqrt{2}$.

Hence this is a *saddle point*. As you can check, the phase portrait near $(1,1)$ is as shown in Figure 6.4.4.

$\Delta = -1 < 0$

$TrJ = -2 < 0$



Eigenvalues:

$\lambda_1 \approx -2.41421$

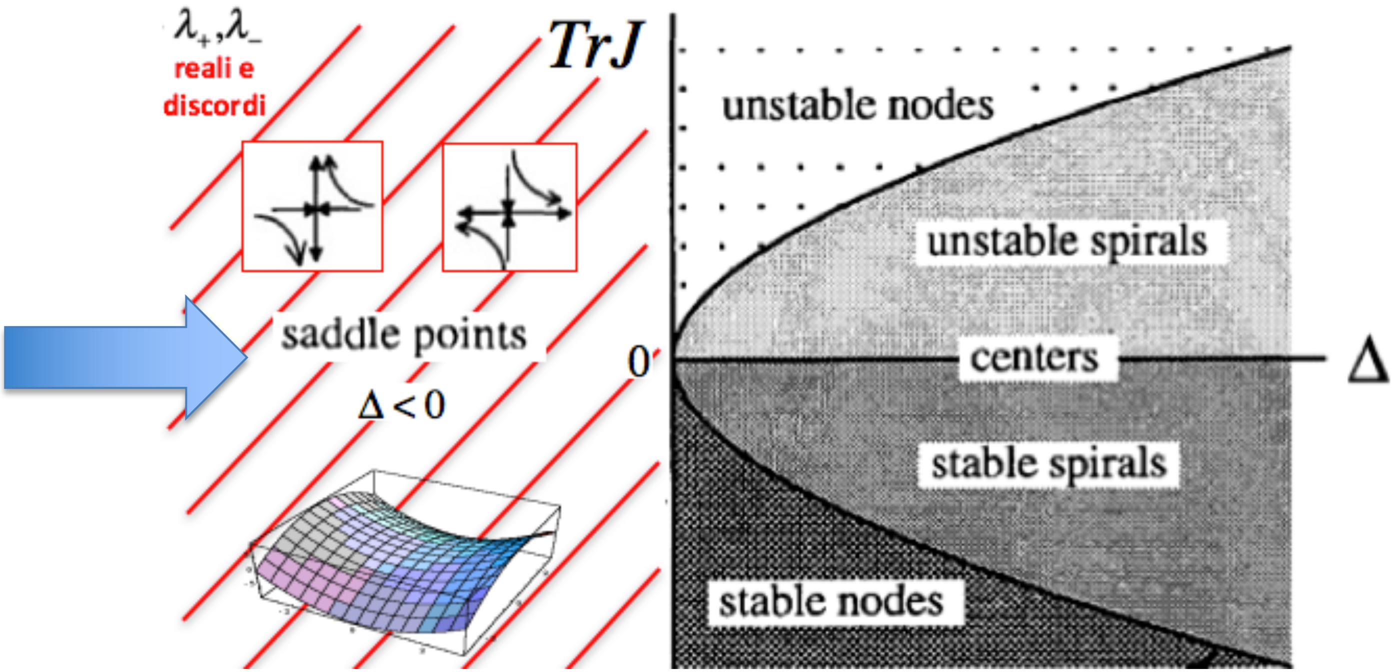
$\lambda_2 \approx 0.414214$

Eigenvectors:

$v_1 = (\sqrt{2}, 1)$

$v_2 = (-\sqrt{2}, 1)$

Figure 6.4.4



Combining Figures 6.4.1–6.4.4, we get Figure 6.4.5, which already conveys a good sense of the entire phase portrait. Furthermore, notice that the x and y axes contain straight-line trajectories, since $\dot{x} = 0$ when $x = 0$, and $\dot{y} = 0$ when $y = 0$.

$$\begin{cases} \dot{x} = x(3 - x - 2y) \\ \dot{y} = y(2 - x - y) \end{cases}$$

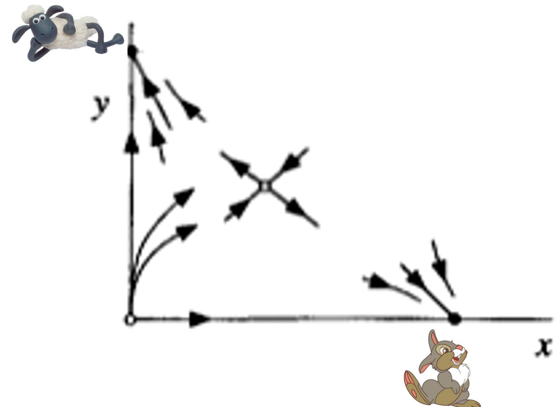
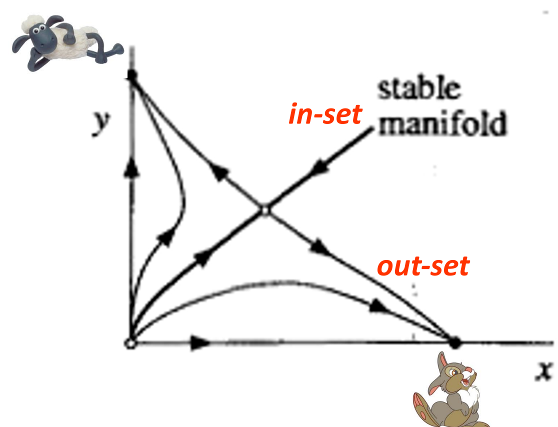


Figure 6.4.5

Now we use common sense to fill in the rest of the phase portrait (Figure 6.4.6). For example, some of the trajectories starting near the origin must go to the stable node on the x-axis, while others must go to the stable node on the y-axis. In between, there must be a special trajectory that can't decide which way to turn, and so it dives into the saddle point. This trajectory is part of the ***stable manifold*** of the saddle, drawn with a heavy line in Figure 6.4.6.



The other branch of the stable manifold consists of a trajectory coming in "from infinity." A computer-generated phase portrait (Figure 6.4.7) confirms our sketch.

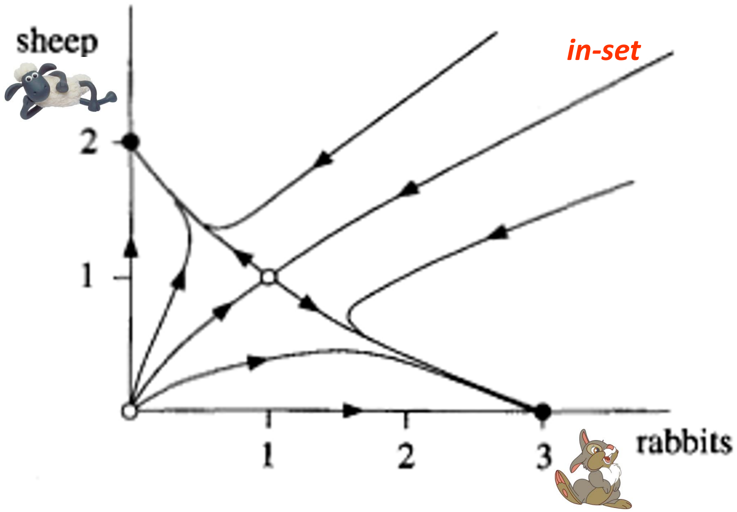
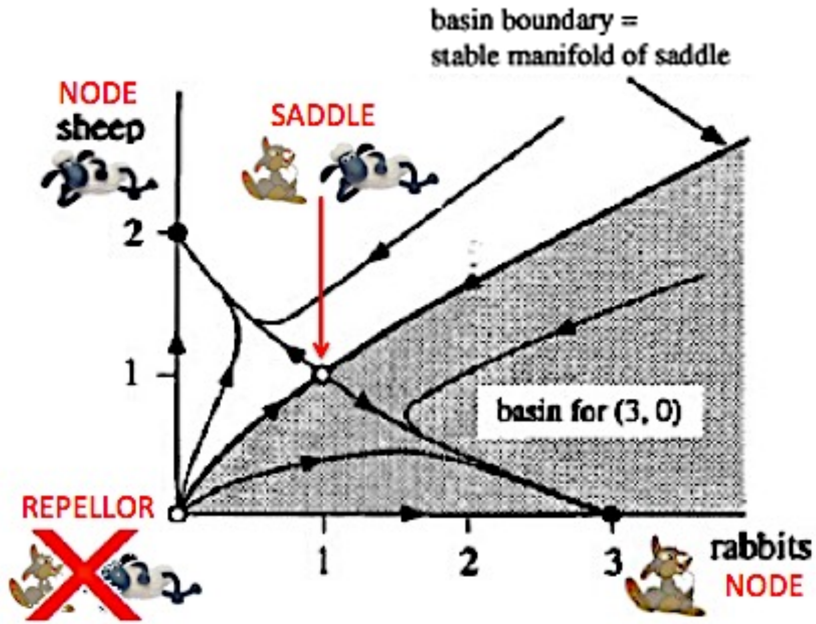


Figure 6.4.7

The phase portrait has an interesting biological interpretation. It shows that one species generally drives the other to extinction. Trajectories starting below the stable manifold lead to eventual extinction of the sheep, while those starting above lead to eventual extinction of the rabbits. This dichotomy occurs in other models of competition and has led biologists to formulate the *principle of competitive exclusion*, which states that two species competing for the same limited resource typically cannot coexist.

Our example also illustrates some general mathematical concepts. Given an attracting fixed point $\mathbf{x}^*$, we define its ***basin of attraction*** to be the set of initial conditions $\mathbf{x}_0$ such that $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$. For instance, the basin of attraction for the node at $(3,0)$ consists of all the points lying below the stable manifold of the saddle. This basin is shown as the shaded region in Figure 6.4.8.



Because the stable manifold separates the basins for the two nodes, it is called the ***basin boundary***. For the same reason, the two trajectories that comprise the stable manifold are traditionally called ***separatrices***. Basins and their boundaries are important because they partition the phase space into regions of different long-term behavior.

rabbits_sheep.nlogo