Classificazione dei Sistemi Dinamici

Sistemi dinamici continui (Flussi)

$$\dot{X} = f(X)$$

Sistemi dinamici discreti (Mappe)

$$x_{n+1} = Ax_n(1 - x_n) \equiv f_A(x)$$

Flussi Dissipativi

Attrattori

1D Punto fisso

2D Ciclo Limite

3D Caotici

Flussi Hamiltoniani

Orbite

Periodiche

Quasi Periodiche

Caotiche

Mappe Dissipative

Attrattori

Punto fisso

Ciclo Limite

Caotici

Mappe Conservative (area-preserving)

Orbite

Periodiche

Quasi Periodiche

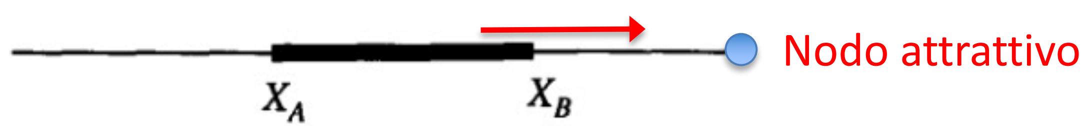
Caotiche

$$\boxed{\dot{X} = f(X)}$$

Flussi dissipativi in una dimensione

$$\frac{1}{L}\frac{dL}{dt} = \frac{1}{L}\left[f(X_B) - f(X_A)\right] = \frac{df(X)}{dX} < 0$$

fixed points (dim.0)



A "cluster of initial conditions," indicated by the heavy line, along the X axis.

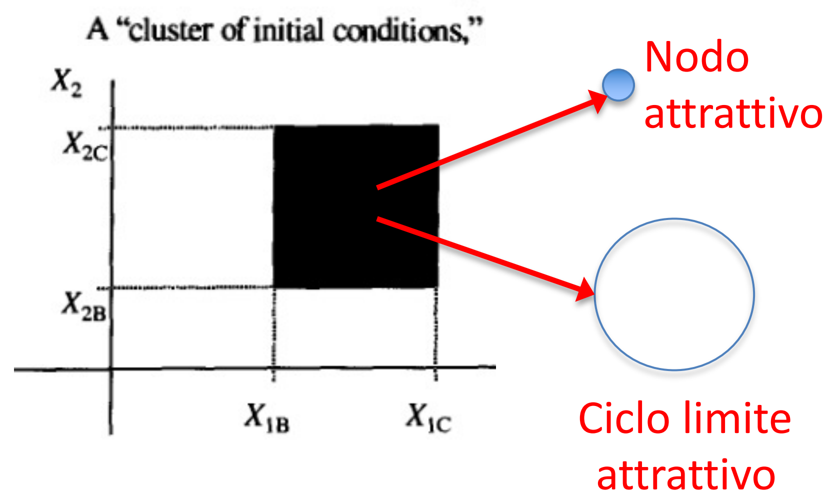
$$\dot{X}_1 = f_1(X_1, X_2)$$
$$\dot{X}_2 = f_2(X_1, X_2)$$

Flussi dissipativi in due dimensioni

$$\frac{1}{A}\frac{dA}{dt} = \frac{\partial f_1}{\partial X_1} + \frac{\partial f_2}{\partial X_2} < 0$$

fixed points (dim.0)

limit cycles (dim.1)



Metodo dello Jacobiano per studiare i punti fissi nel caso generale a 2 dim.

Equazioni originarie

$$\dot{X}_1 = f_1(X_1, X_2)$$
$$\dot{X}_2 = f_2(X_1, X_2)$$

...ricavare i punti fissi...

Equazioni linearizzate nelle vicinanze di un dato punto fisso (X_{1o}, X_{2o})

$$\dot{x}_1 = \frac{\partial f_1}{\partial x_1} x_1 + \frac{\partial f_1}{\partial x_2} x_2$$
$$\dot{x}_2 = \frac{\partial f_2}{\partial x_1} x_1 + \frac{\partial f_2}{\partial x_2} x_2$$

with $f_{ij} = \frac{\partial f_i}{\partial x_j}$

...calcolate nel punto fisso

Distanze dal punto fisso

$$x_1 = X_1 - X_{1o}$$
$$x_2 = X_2 - X_{2o}$$

3.14 The Jacobian Matrix for Characteristic Values

We would now like to introduce a more elegant and general method of finding the characteristic equation for a fixed point. This method makes use of the so-called ***Jacobian matrix*** of the derivatives of the time evolution functions. Once we see how this procedure works, it will be easy to generalize the method, at least in principle, to find characteristic values for fixed points in state spaces of any dimension. The Jacobian matrix for the system is defined to be the following square array of the derivatives:

Matrice Jacobiana

$$J = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \rightarrow \boxed{\lambda_+, \lambda_-} \qquad (3.14\text{-}1)$$

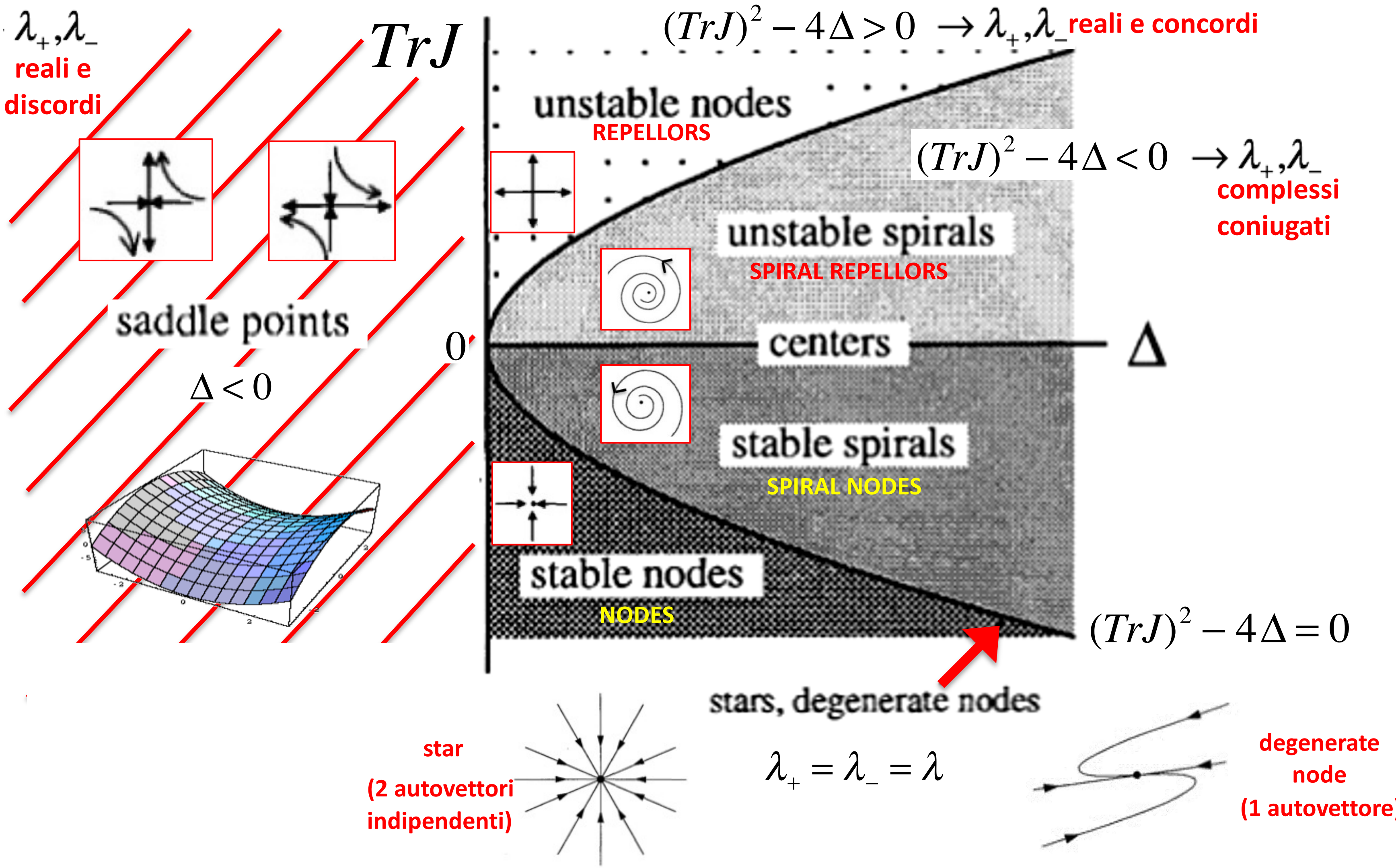
Autovalori

where the derivatives are evaluated at the fixed point. We subtract λ from each of the principal diagonal (upper left to lower right) elements and set the determinant of the matrix equal to 0:

Diagramma dei Punti Fissi in uno Spazio degli Stati a Due Dimensioni

$$\lambda_{\pm} = \frac{TrJ \pm \sqrt{(TrJ)^2 - 4\Delta}}{2}$$

con: $\begin{cases} TrJ = f_{11} + f_{22} \\ \Delta = f_{11}f_{22} - f_{21}f_{12} \end{cases}$



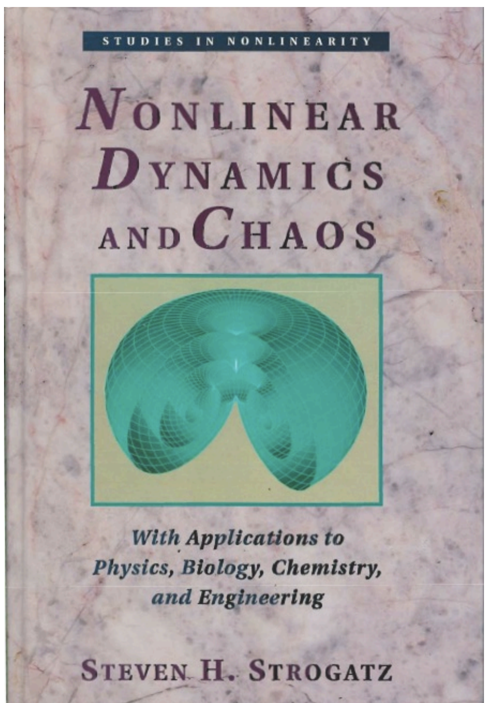
Steven Strogatz

Sheep

Rabbit

6.4 Rabbits versus Sheep

In the next few sections we'll consider some simple examples of phase plane analysis. We begin with the classic ***Lotka–Volterra model of competition*** between two species, here imagined to be rabbits and sheep. Suppose that both species are competing for the same food supply (grass) and the amount available is limited. Furthermore, ignore all other complications, like predators, seasonal effects, and other sources of food. Then there are two main effects we should consider:



Rabbit

Steven Strogatz

Sheep

6.4 Rabbits versus Sheep

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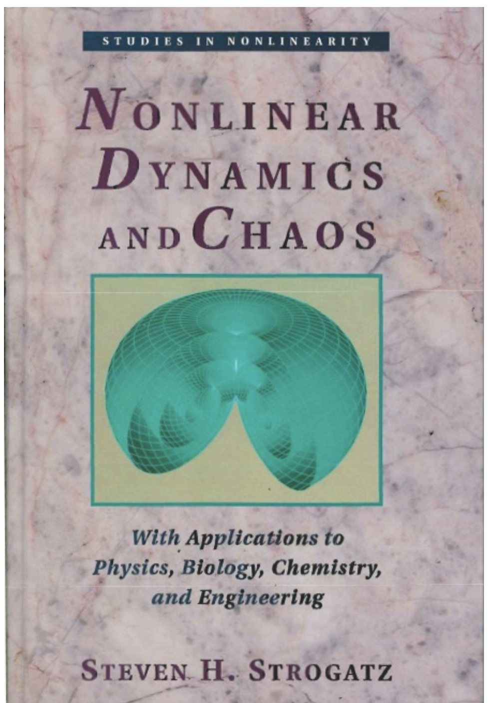
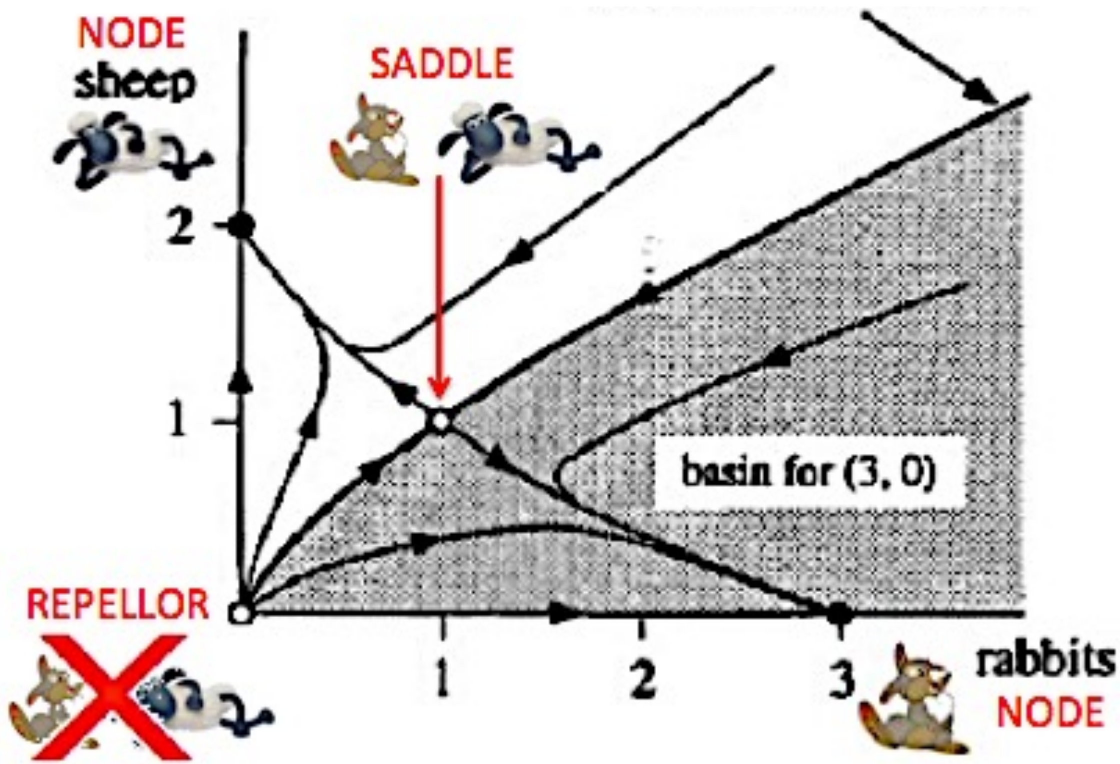
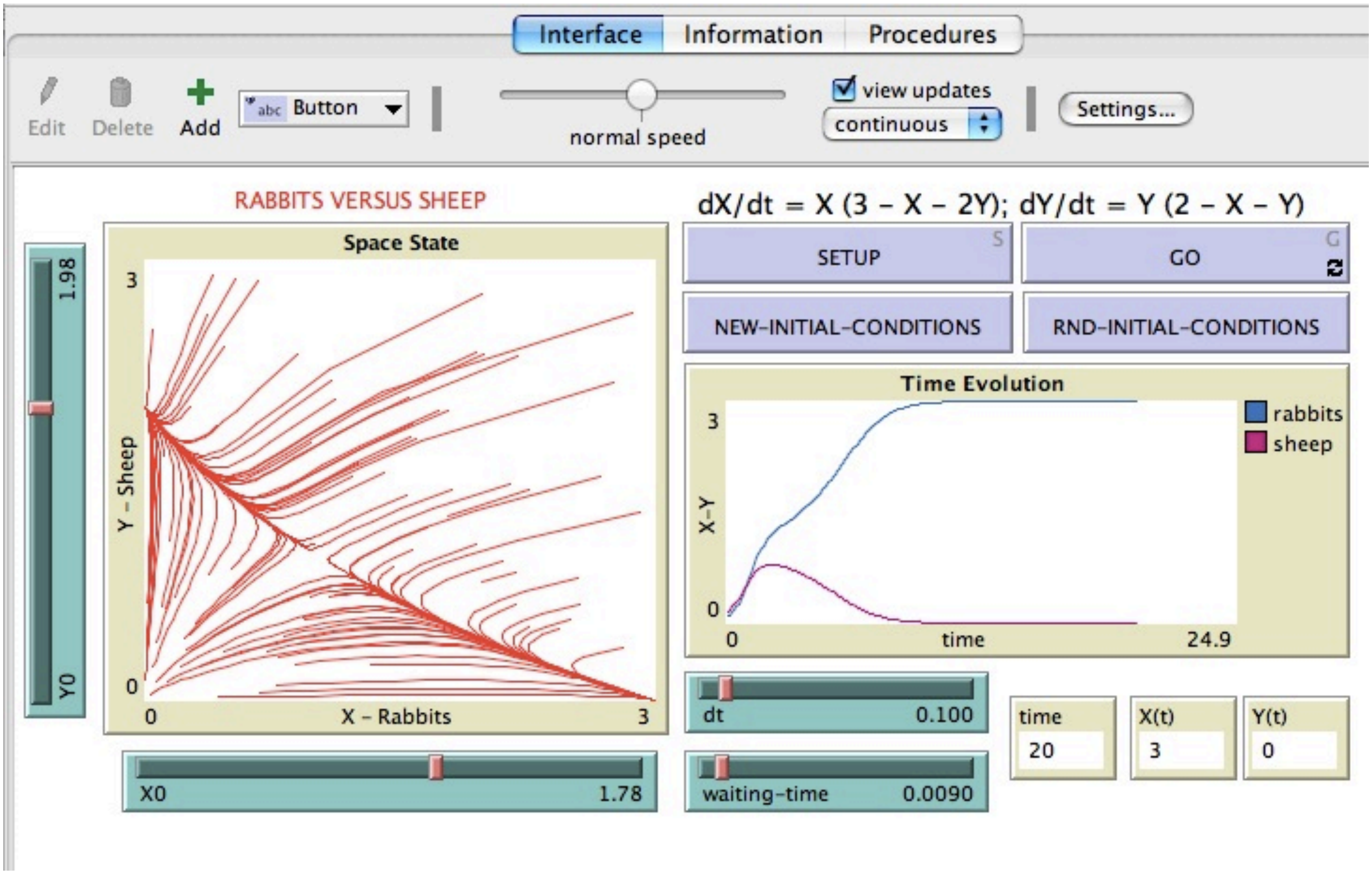




rabbits_sheep.nlogo



Punti fissi con valori caratteristici complessi coniugati

Example: The Brusselator Model

As an illustration of our techniques, let us return to the Brusselator Model given in Eq. (3.11-1).

The Brusselator Model

$$\boxed{\begin{aligned} \dot{X} &= A-(B+1)X + X^2Y \\ \dot{Y} &= BX - X^2Y \end{aligned}} \qquad (3.11\text{-}1)$$

First let us find the fixed points for this set of equations. By setting the time derivatives equal to 0, we find that the fixed points occur at the values X,Y that satisfy

$$\begin{cases} A-(B+1)X + X^2Y = 0 & (3.11\text{-}2) \\ BX - X^2Y = 0 & (3.11\text{-}3) \end{cases}$$

We see that there is just one point (X,Y) which satisfies these equations, and the coordinates of that fixed point are $X_0 = A$, $Y_0 = B/A$.

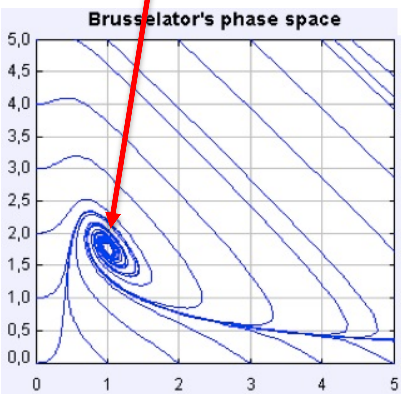
Ilya Prigogine
(1917-2003)

Punti fissi con valori caratteristici complessi coniugati

$$\dot{X} = A - (B+1)X + X^2Y$$
$$\dot{Y} = BX - X^2Y$$

1 punto fisso:

$X_0 = A,\ Y_0 = B/A.$



Ilya Prigogine (1917-2003)

The Jacobian matrix for that set of equations is

$$J = \begin{pmatrix} (B-1) & A^2 \\ -B & -A^2 \end{pmatrix} \qquad \Delta = A^2 \qquad TrJ = (B-1) - A^2 \tag{3.14-7}$$

Following the Jacobian determinant method outlined earlier, we find the characteristic values:

$$\lambda_\pm = \frac{TrJ \pm \sqrt{(TrJ)^2 - 4\Delta}}{2} \rightarrow \lambda_\pm = \frac{1}{2}\left[(B-1) - A^2\right] \pm \frac{1}{2}\sqrt{\left(A^2 - (B-1)\right)^2 - 4A^2} \tag{3.14-8}$$

$$\Delta = 1$$
$$TrJ = B - 2$$

In the discussion of this model, it is traditional to set $\boxed{A = 1}$ and let B be the control parameter. Let us follow that tradition. We see that with $\boxed{B < 2}$, both characteristic values have negative real parts and the fixed point is a spiral node. This result tells us that the chemical concentrations tend toward the fixed point values $X_0 = A = 1$, $Y_0 = B$ as time goes on. They oscillate, however, with the frequency $\Omega = |B(B-4)|^{1/2}$ as they head toward the attractor. For $\boxed{2 < B < 4}$, the fixed point becomes a spiral repellor. However, our analysis cannot tell us what happens to the trajectories as they spiral away from the fixed point. As we shall learn in the next section, they tend to a limit cycle as shown in Fig. I.1 in Section I (for a different model).

Ex: A=1,B=1 → Δ=1, TrJ=-1, $TrJ^2-4\Delta<0$: Spiral Node

A=1,B=3 → Δ=1, TrJ= 1, $TrJ^2-4\Delta<0$: Spiral Repellor

Brussellator.nlogo

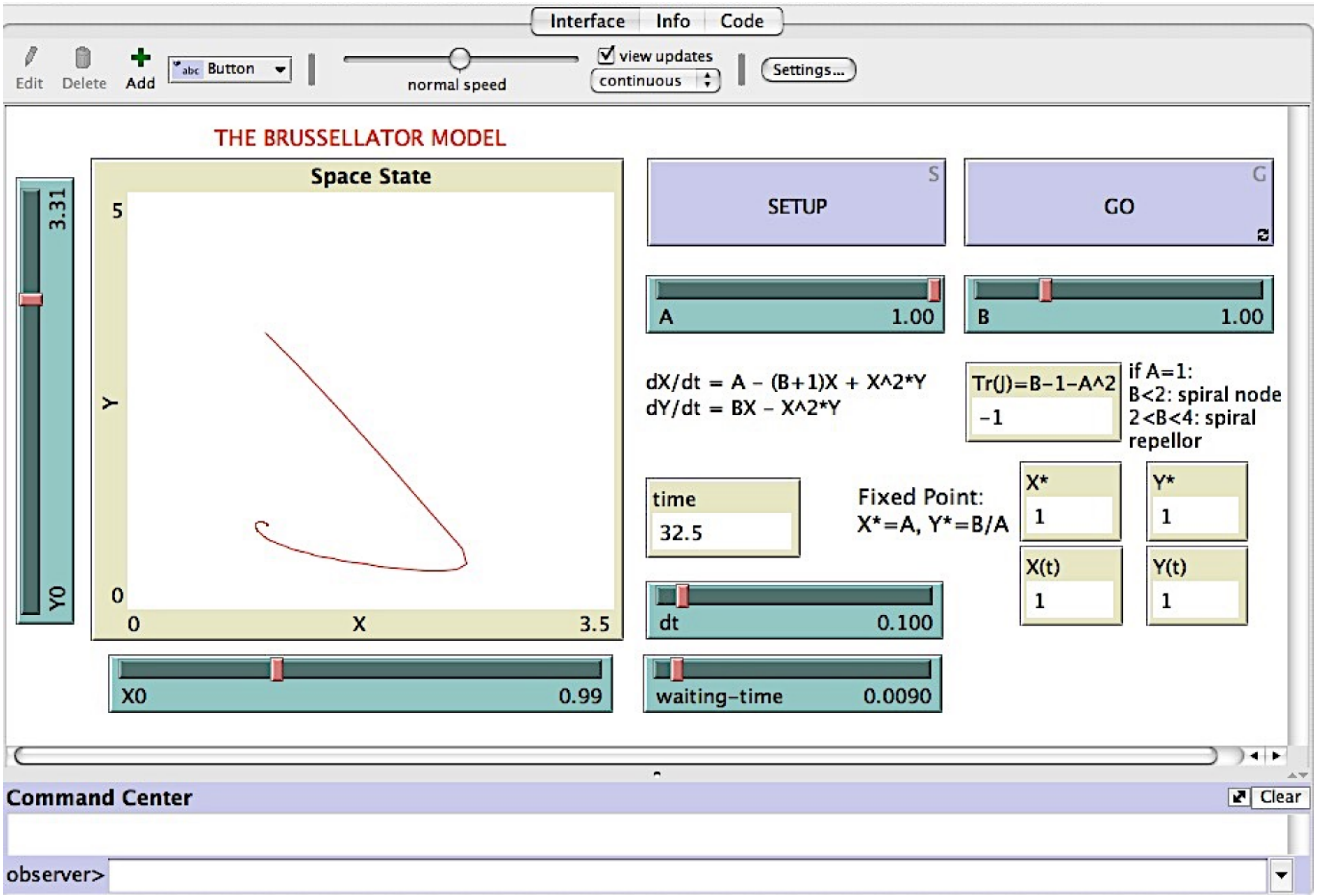
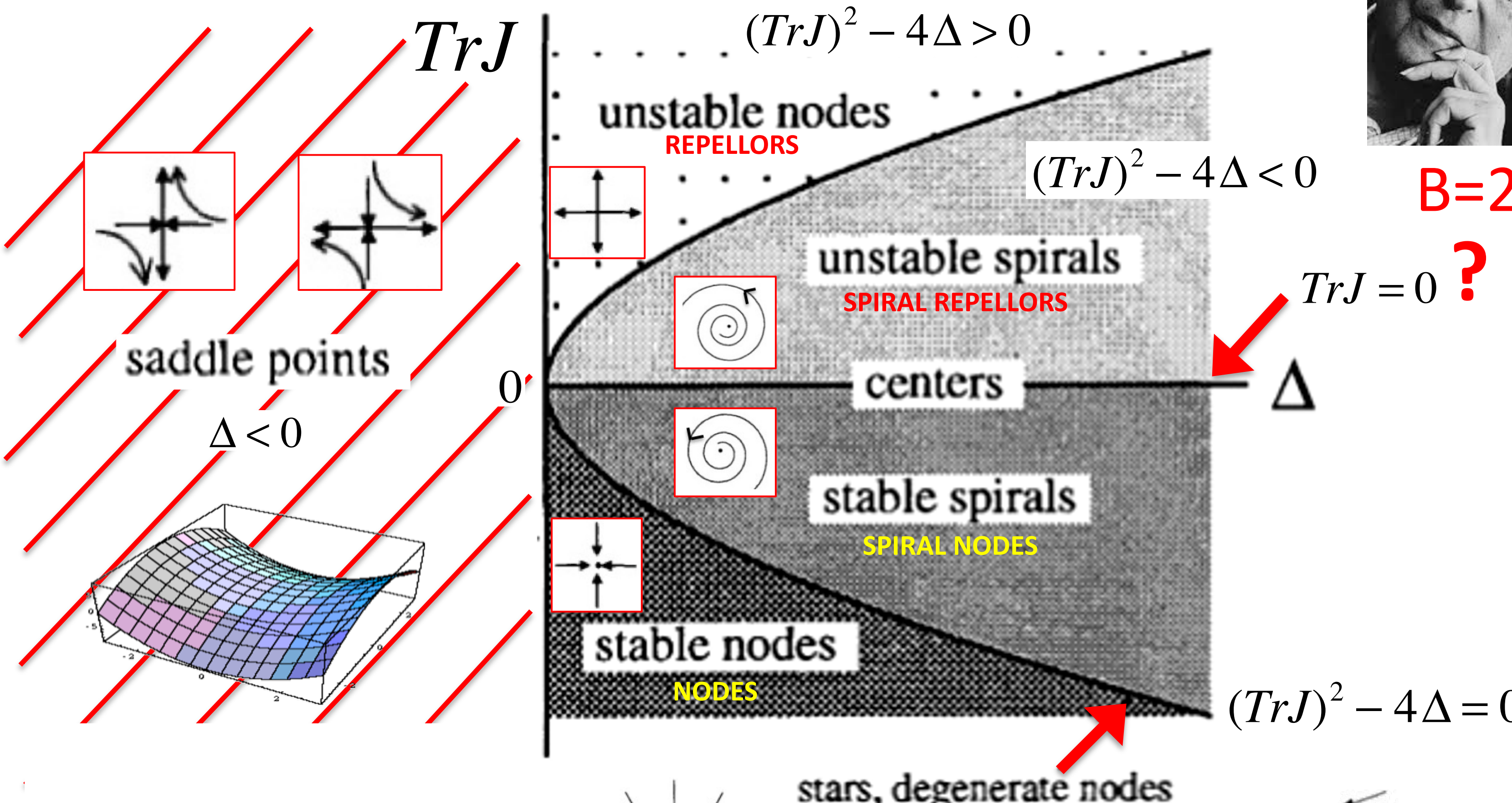
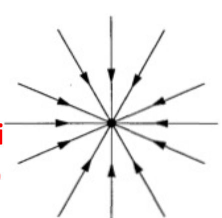


Diagramma dei Punti Fissi in uno Spazio degli Stati a Due Dimensioni

$$\lambda_{\pm} = \frac{TrJ \pm \sqrt{(TrJ)^2 - 4\Delta}}{2}$$



star
(2 autovettori indipendenti)



$$\lambda_+ = \lambda_- = \lambda$$

degenerate node
(1 autovettore)

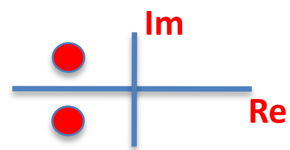
3.15 Limit Cycles

In state spaces with two or more dimensions, it is possible to have cyclic or periodic behavior. This very important kind of behavior is represented by closed loop trajectories in the state space. A trajectory point on one of these loops continues to cycle around that loop for all time. These loops are called ***limit cycles*** if the cycle is isolated, that is if trajectories nearby either approach or are repelled from the limit cycle. The discussion in the previous section indicated that motion on a limit cycle in state space represents oscillatory, repeating motion of the system. The oscillatory behavior is of crucial importance in many practical applications, ranging from radios to brain waves.

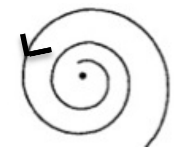
$$(TrJ)^2 - 4\Delta < 0 \quad \rightarrow \lambda_+, \lambda_-$$

complessi
coniugati

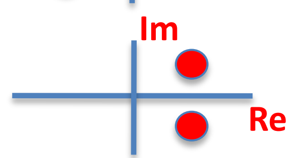
$TrJ < 0$



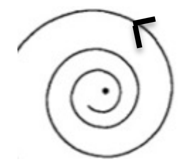
SPIRAL
NODE



$TrJ > 0$

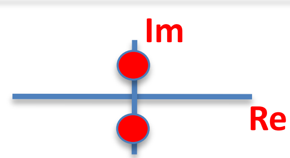


SPIRAL
REPELLOR

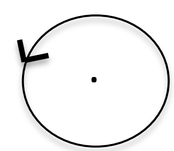


$$x_1(t) = F_1 e^{Rt} \sin\Omega t$$
$$x_2(t) = F_2 e^{Rt} \sin\Omega t$$

$$R = TrJ = 0$$



CICLO
LIMITE!



Il Teorema di Poincaré-Bendixson

We shall formulate the analysis in answer to two questions: (1) When do limit cycles occur? and (2) When is a limit cycle stable or unstable? The first question is answered for a two-dimension state space by the famous ***Poincaré–Bendixson Theorem***. The theorem can be formulated in the following way:

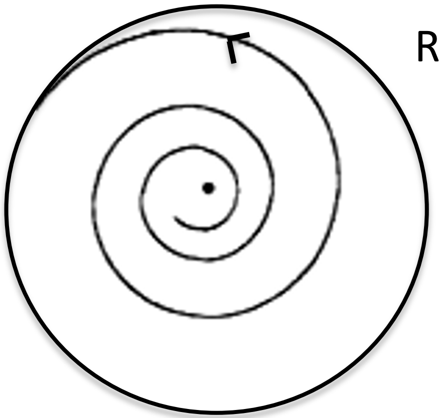
1. Suppose the long-term motion of a state point in a two-dimensional state space is limited to some finite-size region; that is, the system doesn't wander off to infinity.
2. Suppose that this region (call it *R*) is such that any trajectory starting within *R* stays within *R* for all time. [*R* is called an "invariant set" for that system.]
3. Consider a particular trajectory starting in *R*. The Poincaré–Bendixson Theorem states that there are only two possibilities for that trajectory:

a. The trajectory approaches a fixed point of the system as $t \to \infty$.
b. The trajectory approaches a limit cycle as $t \to \infty$.

A proof of this theorem is beyond the scope of this book. The interested reader is referred to [Hirsch and Smale, 1974]. We can see, however, that the results are entirely reasonable if we take into account the No-Intersection Theorem and the assumption of a bounded region of state space in which the trajectories live. The reader is urged to draw some pictures of state space trajectories in two dimensions to see that these two principles guarantee that the only two possibilities are fixed points and limit cycles.

Il Teorema di Poincaré-Bendixson

The Brussellator model displays the typical situation in which a limit cycle develops. An invariant region R contains a repelling fixed point. Trajectories starting near the repelling fixed point are pushed away and (if there is no attracting fixed point in R) must head toward a limit cycle (which can be proved to enclose the repellor).



It is important to note that the Poincaré–Bendixson Theorem works only in two dimensions because only in two dimensions does a closed curve separate the space into a region "inside" the curve and a region "outside." Thus a trajectory starting inside the limit cycle can never get out and a trajectory starting outside can never get in. This argument is an excellent example of the power of topological arguments in the study of dynamical systems. Further, from the Poincaré–Bendixson Theorem we arrive at an important result: Chaotic trajectories (in a bounded system) cannot occur in a state space of two dimensions. *For systems described by differential equations, we need at least three state-space dimensions for chaos.*

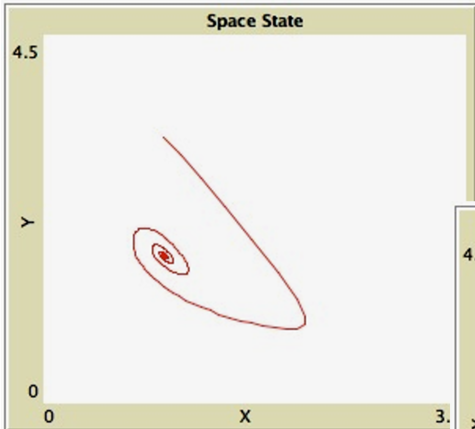
Brussellator.nlogo

$$\dot{X} = A-(B+1)X + X^2Y$$
$$\dot{Y} = BX - X^2Y$$

1 punto fisso:

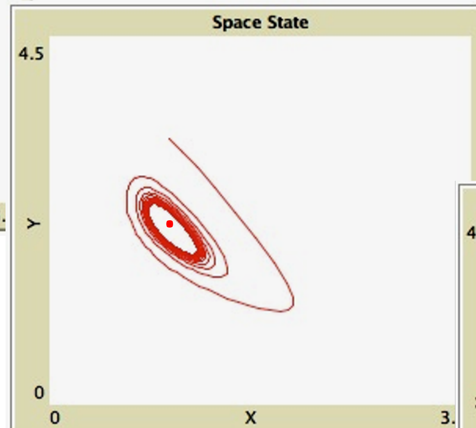
$$X_0 = A,\ Y_0 = B/A.$$

A=1, B=1.80



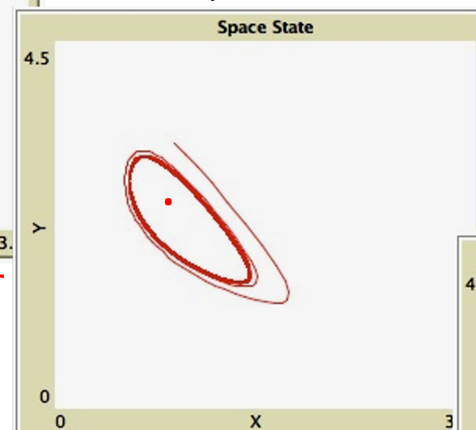
1 stable spiral node

A=1, B=2.15



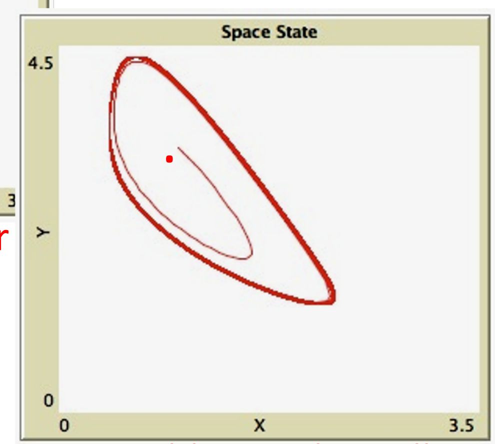
1 unstable spiral repellor
+
1 stable limit cycle

A=1, B=2.33



1 unstable spiral repellor
+
1 stable limit cycle

A=1, B=2.85



1 unstable spiral repellor
+
1 stable limit cycle

A=1,B<2 : stable spiral node

A=1, B=2 : Nasce il ciclo limite!!

A=1, B>2 :
unstable spiral repellor + 1 stable limit cycle

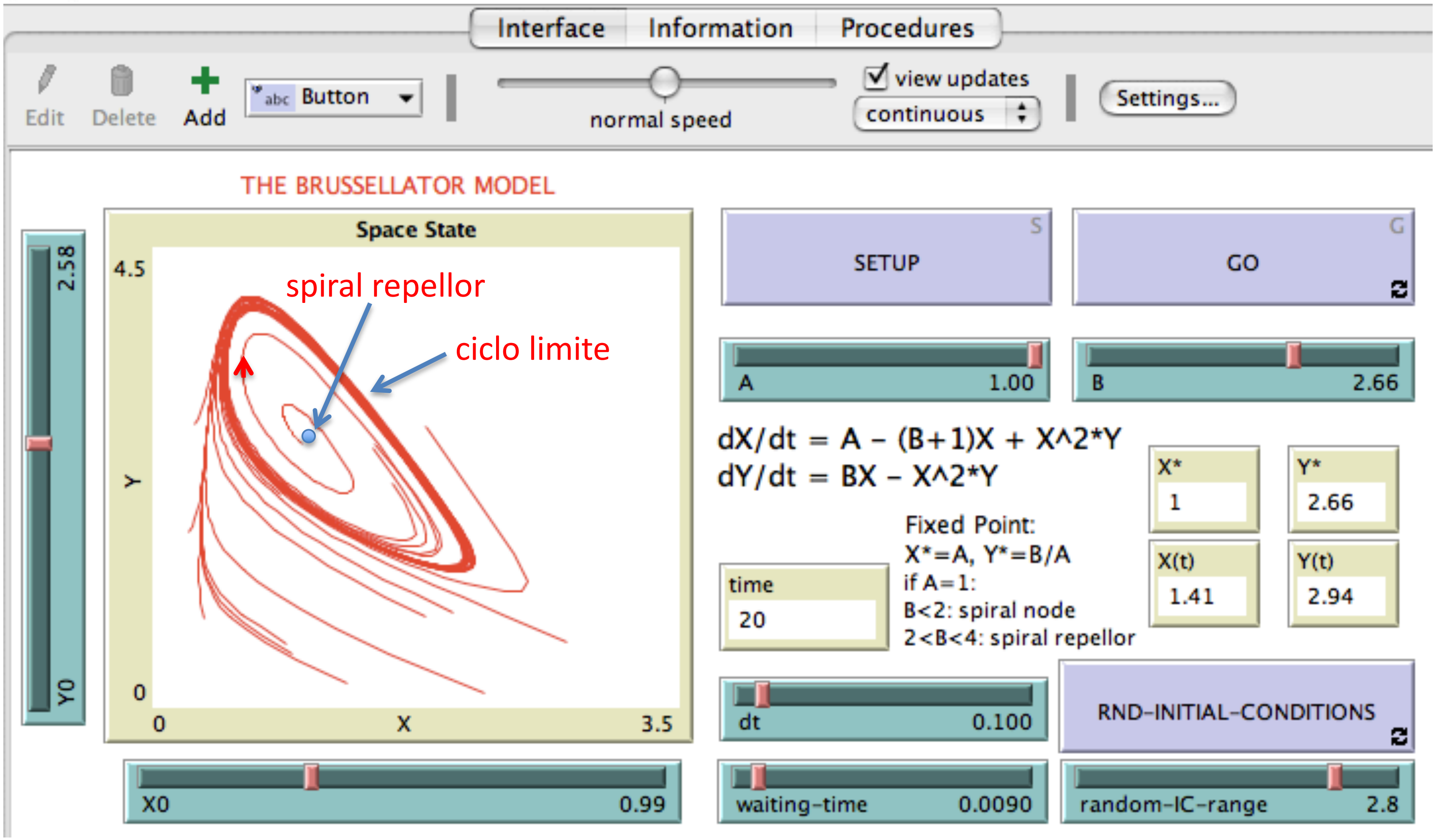
Brussellator-v2.nlogo

The Brussellator model displays the typical situation in which a limit cycle develops. An invariant region R contains a repelling fixed point. Trajectories starting near the repelling fixed point are pushed away and (if there is no attracting fixed point in R) must head toward a limit cycle (which can be proved to enclose the repellor).

$$\dot{X} = A - (B+1)X + X^2Y$$
$$\dot{Y} = BX - X^2Y$$

1 punto fisso:

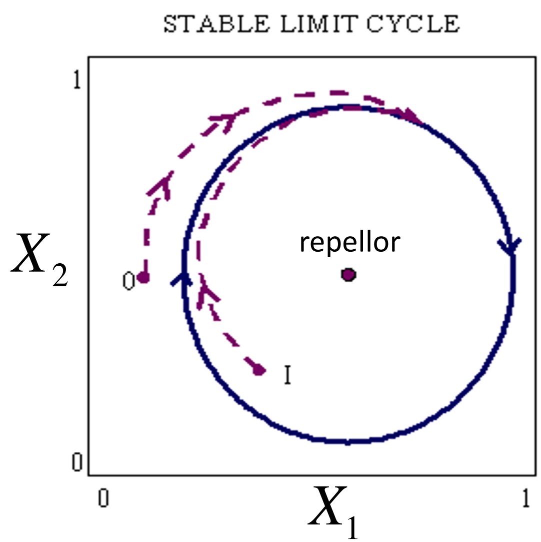
$X_0 = A,\ Y_0 = B/A.$

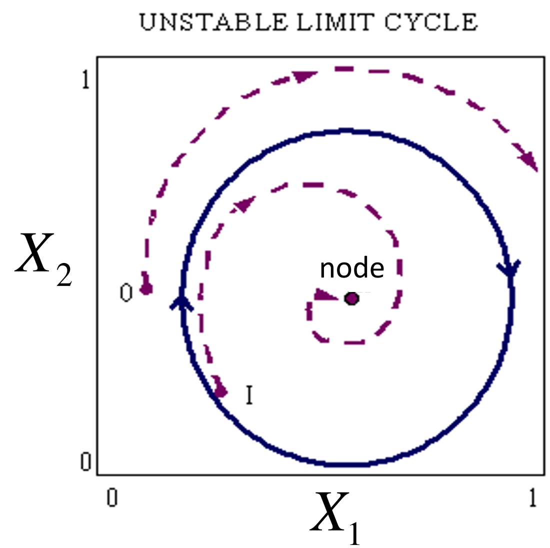


3.16 Poincaré Sections and the Stability of Limit Cycles

We have seen that in state spaces of two (or more) dimensions, a new type of behavior can arise: motion on a limit cycle. The obvious question is the following: Is the motion on the limit cycle stable? That is, if we push the system slightly away from the limit cycle, does it return to the limit cycle (at least asymptotically) or is it repelled from the limit cycle? As we shall see, both possibilities occur in actual systems.

You might expect that we would proceed much as we did for nodes and repellors, by calculating characteristic values involving derivatives of the functions describing the state space evolution. In principle, one could do this, but Poincaré showed that an algebraically and conceptually much simpler method suffices. This method uses what is called a ***Poincaré section*** of the limit cycle. The Poincaré section is closely related to the stroboscopic portraits used in Chapter 1 to discuss the behavior of the diode circuit.





Costruzione della Sezione di Poincaré

For a two-dimensional state space, the Poincaré section is constructed as follows. In the two-dimensional state space, we draw a line segment that cuts through the limit cycle as shown in Fig. 3.12 (a). This line can be any line segment, but in some cases one might wish to choose the X_1 or X_2 axes. Let us call the point at which the limit cycle crosses the line segment going, say, point P.

J.H.Poncaré (1854-1912)

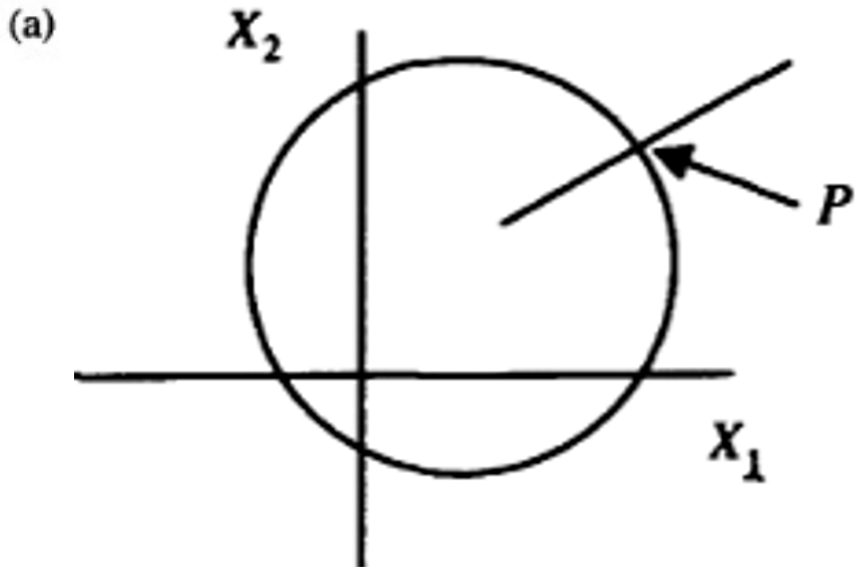
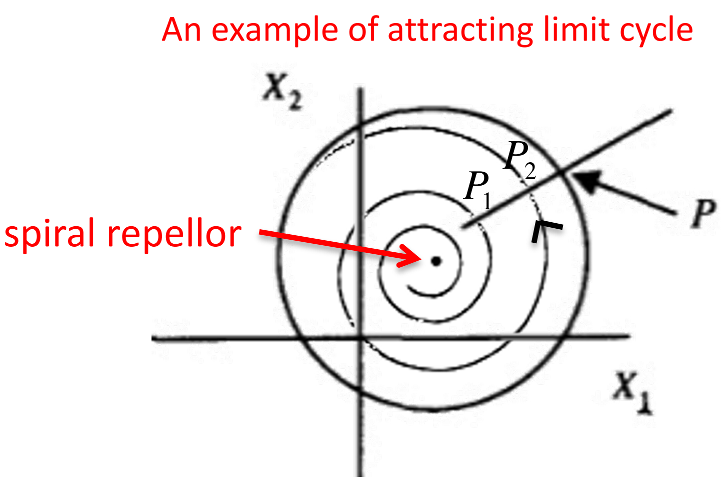


Fig. 3.12. (a) The Poincaré line segment intersects the limit cycle at point P. (b) The four possibilities for sequences of Poincaré intersection points for trajectories near a limit cycle in two dimensions.

If we now start a trajectory in the state space at a point that is close to, but not on, the limit cycle, then that trajectory will cross the Poincaré section line segment at a point other than P. Let's call the first crossing point P_1. As the trajectory evolves, it will cross the Poincaré line segment again at points P_2, P_3, and so on. If the sequence of points approaches P as time goes on for any starting point in the neighborhood of the limit cycle, we say that we have an ***attracting limit cycle*** or, equivalently, a ***stable limit cycle***. In other words, the limit cycle is an attractor for the system. If the sequence of intersection points moves away from P (for any trajectory starting near the limit cycle), we say we have a ***repelling limit cycle*** or, equivalently, an ***unstable limit cycle***. Another possibility is that the points are attracted on one side and repelled on the other: In that case we say that we have a ***saddle cycle*** (in analogy with a saddle point). These possibilities are shown graphically in Fig. 3.12 (b).



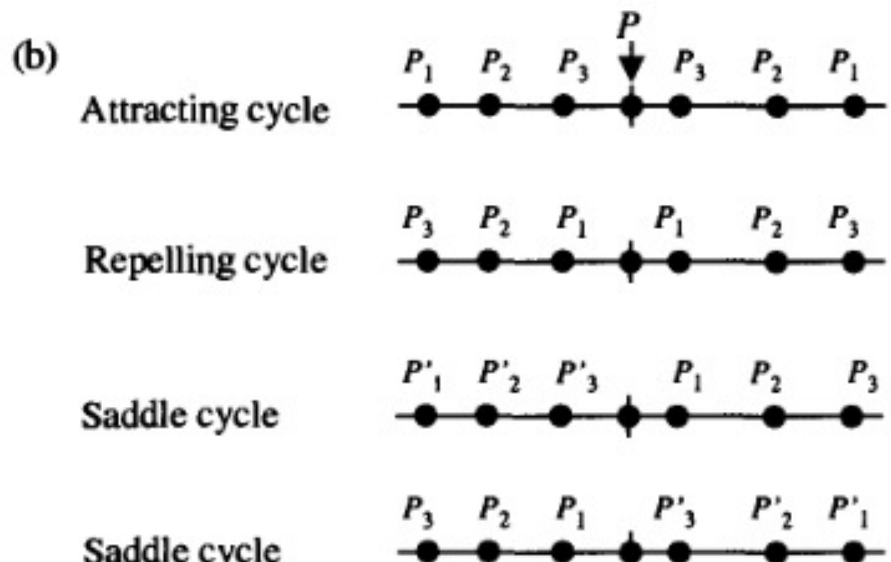


Fig. 3.12. (a) The Poincaré line segment intersects the limit cycle at point P. (b) The four possibilities for sequences of Poincaré intersection points for trajectories near a limit cycle in two dimensions.

How do we describe these properties quantitatively? We use what is called a ***Poincaré map function*** (or ***Poincaré map***, for short). The essential idea is that given a point P_1, where a trajectory crosses the Poincaré line segment, we can in principle determine the next crossing point P_2 by integrating the time-evolution equations describing the system. So, there must be some mathematical function, call it F, that relates P_1 to P_2: $P_2 = F(P_1)$. (Of course, finding this function F is equivalent to solving the original set of equations and that may be difficult or impossible in actual practice.) In general, we may write

$$\boxed{P_{n+1} = F(P_n)} \tag{3.16-1}$$

In general the function F depends not only on the original equations describing the system, but on the choice of the Poincaré line segment as well.

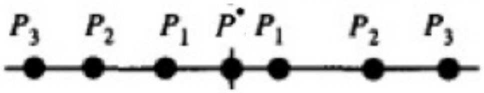
To analyze the nature of the limit cycle, we can analyze the nature of the function F and its derivatives. Two points are important to notice:

2 Vantaggi

1. The Poincaré section reduces the original two-dimensional problem to a one-dimensional problem.
2. The Poincaré map function states an iterative (finite-size time step) relation rather than a differential (infinitesimal time step) relation.

The last point is important because F gives P_{n+1} in terms of P_n. The time interval between these points is roughly the time to go around the limit cycle once, a relatively big jump in time. On the other hand, a one-dimensional differential equation $\dot{x} = f(x)$ tells us how x changes over an infinitesimal time interval. The function F is sometimes called an *iterated map function* (or *iterated map*, for short). (Because of the importance of iterated maps in nonlinear dynamics, we shall devote Chapter 5 to a study of their properties.)

Let us note that the point P on the limit cycle satisfies $P = F(P)$. Any point P^* that satisfies $P^* = F(P^*)$ is called a *fixed point* of the map function. If a trajectory crosses the line segment exactly at P^*, it returns to P^* on every cycle. In analogy with our discussion of fixed points for differential equations, we can ask what happens to a point P_1 close to P^*. In particular, we ask what happens to the distance between P_1 and P^* as the system evolves. Formally, we look at



$$P_2 - P^* = \boxed{F(P_1)} - F(P^*) \tag{3.16-2}$$

and use a Taylor series expansion about the point P^* to write

$$P_2 - P^* = \boxed{F(P^*) + \left.\frac{dF}{dP}\right|_{P^*} (P_1 - P^*) + \ldots} - F(P^*) \tag{3.16-3}$$

If we define $d_i = (P_i - P^*)$, we see that

$$d_2 = \left.\frac{dF}{dP}\right|_{P^*} d_1 \tag{3.16-4}$$

We now define the ***characteristic multiplier*** *M* for the Poincaré map:

$$M = \left.\frac{dF}{dP}\right|_{P^*} \qquad \textbf{(M>0)} \tag{3.16-5}$$

M is also called the ***Floquet multipler*** or the ***Lyapunov multiplier***. In terms of *M*, we can write Eq. (3.16-4)

$$d_2 = Md_1 \tag{3.16-6}$$

We find in general

$$d_{n+1} = M^n d_1 \tag{3.16-7}$$

$$d_{n+1} = M^n d_1 \tag{3.16-7}$$

We see that if $M < 1$, then $d_2 < d_1$, $d_3 < d_2$, and so on: The intersection points approach the fixed point P. In that case the cycle is an attracting limit cycle. If $M > 1$, then the distances grow with repeated iterations, and the limit cycle is a repelling cycle. For saddle cycles, M is equal to 1 but the derivative of the map function is greater than 1 on one side of the cycle and less than 1 on the other side. However, based on our discussion of saddle points for one-dimensional state spaces, we expect that saddle cycles are rare in two-dimensional state spaces. Table 3.4 lists the possibilities.

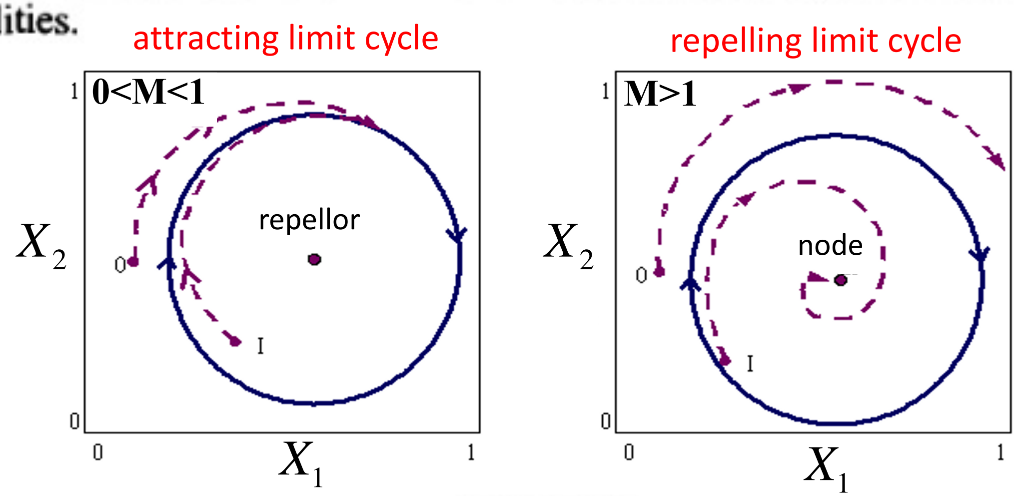


Table 3.4.
The Possible Limit Cycles and Their Characteristic Multipliers for Two–Dimensional State Space

Characteristic Multiplier	Type of Cycle
M < 1	Attracting Cycle
M > 1	Repelling Cycle
M = 1	Saddle Cycle (rare in two-dimensions)

Ma… cosa possiamo dire sul caso M<0?

We can also define a ***characteristic exponent*** associated with the cycle by the equation

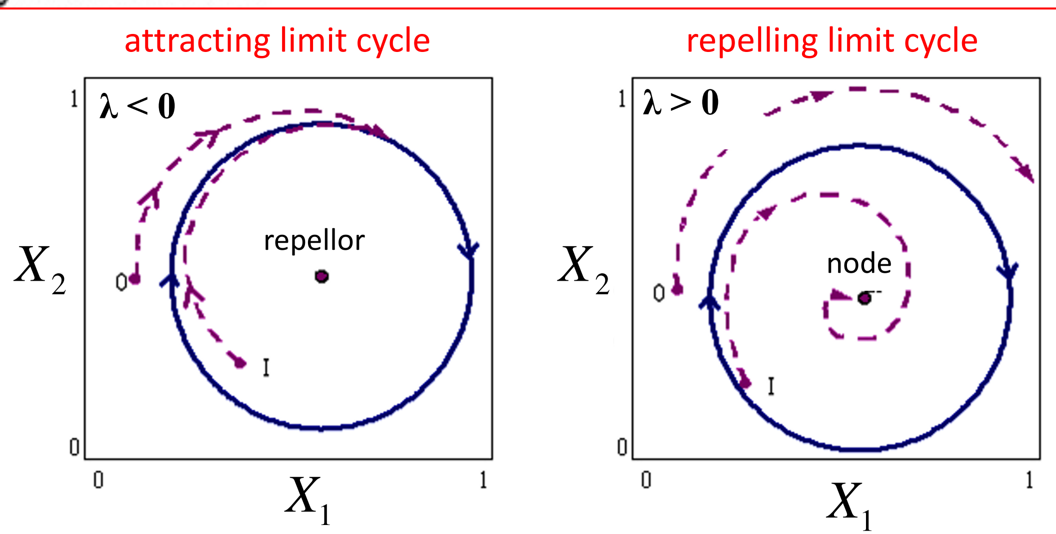
$$M \equiv e^{\lambda} \tag{3.16-8}$$

or

$$\lambda \equiv \ln(M) \tag{3.16-9}$$

The idea is that the characteristic exponent plays the role of the Lyapunov exponent but the time unit is taken to be the time from one crossing of the Poincaré section to the next.

Let us summarize: The Poincaré section method allows us to characterize the possible types of limit cycles and to recognize the kinds of changes that take place in those limit cycles. However, in most cases, we cannot find the mapping function F explicitly; therefore, our ability to predict the kinds of limit cycles that occur for a given system is limited.



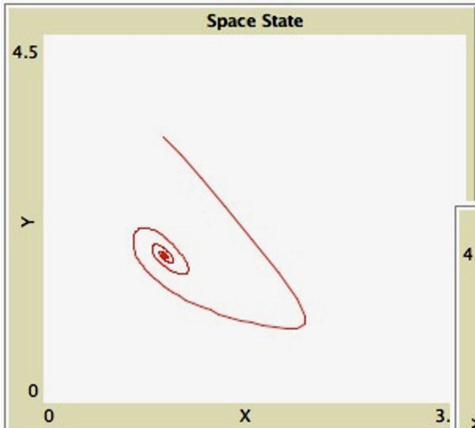
Brussellator.nlogo

$$\dot{X} = A - (B+1)X + X^2Y$$
$$\dot{Y} = BX - X^2Y$$

1 punto fisso:

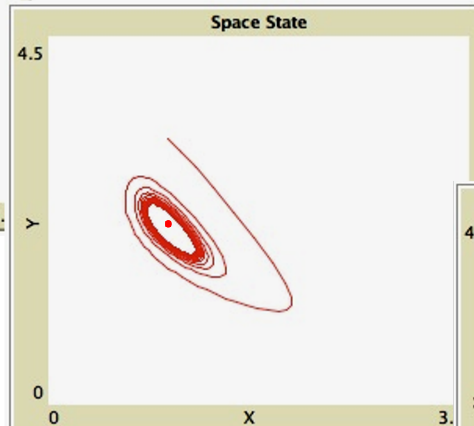
$$X_0 = A,\ Y_0 = B/A.$$

A=1, B=1.80



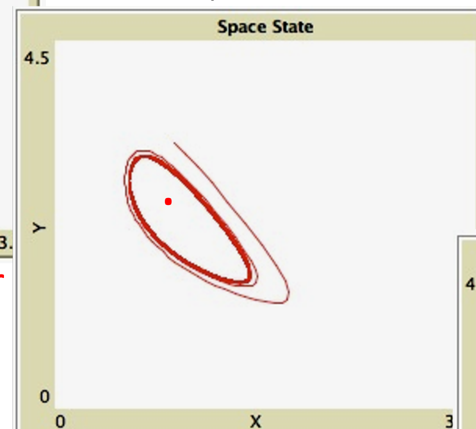
1 stable spiral node

A=1, B=2.15



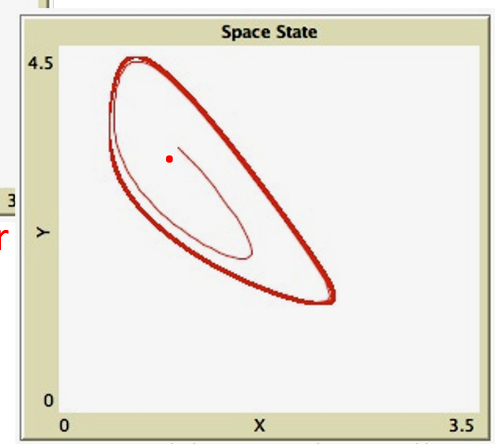
1 unstable spiral repellor
+
1 stable limit cycle

A=1, B=2.33



1 unstable spiral repellor
+
1 stable limit cycle

A=1, B=2.85



1 unstable spiral repellor
+
1 stable limit cycle

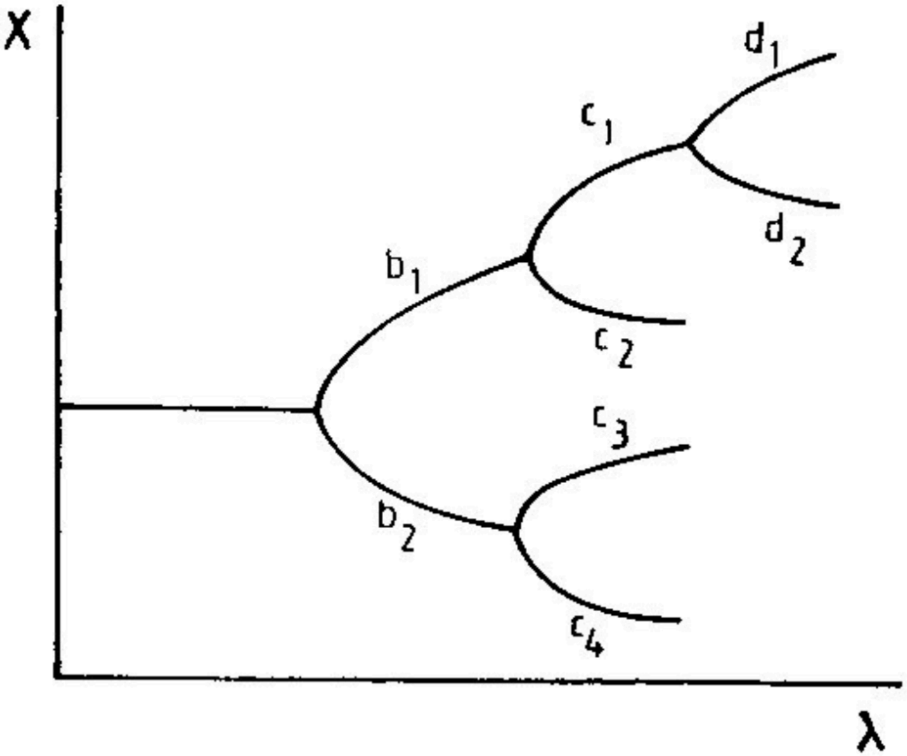
A=1,B<2 : stable spiral node

A=1, B>2 :
unstable spiral repellor + 1 stable limit cycle

A=1, B=2 : BIFORCAZIONE!

Biforcazioni

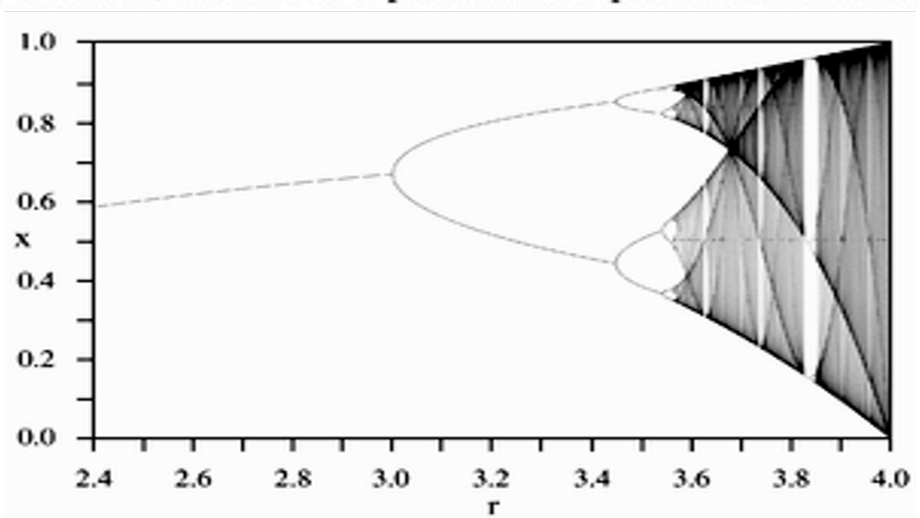
Ilya Prigogine
(1917-2003)



Sequenza di biforcazioni nei sistemi lontani dall'equilibrio

3.17 Bifurcation Theory

We have seen that the characteristic values associated with a fixed point depend on the various parameters used to describe the system. As the parameters change, for example as we adjust a voltage in a circuit or the concentration of chemicals in a reactor, the nature of the characteristic values and hence the character of the fixed point may change. For example, an attracting node may become a repellor or a saddle point. The study of how the character of fixed points (and other types of state space attractors) change as parameters of the system change is called ***bifurcation theory***. (Recall that the term *bifurcation* is used to describe any sudden change in the dynamics of the system. When a fixed point changes character as parameter values change, the behavior of trajectories in the neighborhood of that fixed point will change. Hence the term bifurcation is appropriate here.) Being able to classify and understand the various possible bifurcations is an important part of the study of nonlinear dynamics. However, the theory, as it is presently developed, is rather limited in its ability to predict the kinds of bifurcations that will occur and the parameter values at which the bifurcations take place for a particular system. Description, however, is the first step toward comprehension and understanding.



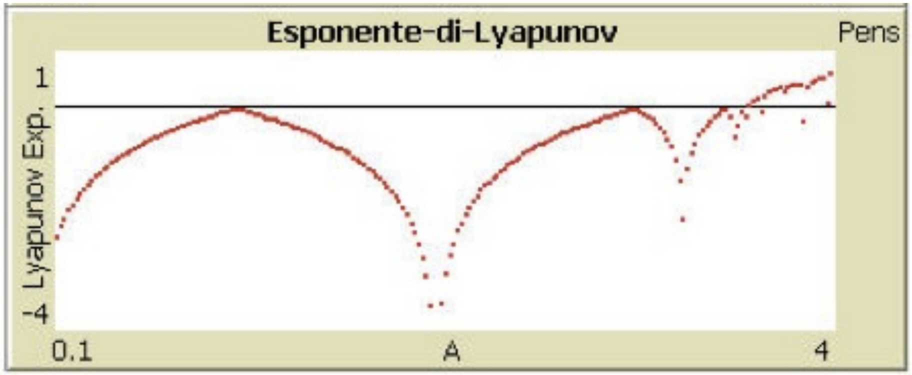
We should also emphasize that simple bifurcation theory treats only the changes in stability of a particular attractor (or, as we shall see in Chapter 4, a particular basin of attraction). Since in general a system may have, for fixed parameter values, several attractors in different parts of state space, we often need to consider the overall dynamical system (that is, its "global" properties) to see what happens to trajectories when a bifurcation occurs.

To keep track of what is happening as the control parameter is varied, we will use two types of diagrams. One type, which we have seen before, is the bifurcation
1 diagram, in which we plot the location of the fixed point (or points) as a function of the control parameter. In the second type of diagram, we plot the characteristic
2 values of the fixed point as a function of the control parameter.

To see how this kind of analysis proceeds, let us begin with the one-dimensional state space case. In a one-dimensional state space, a fixed point has just one characteristic value λ. The crucial assumption in the analysis is that λ varies smoothly (continuously) as some parameter, call it μ, varies. For example, if $\lambda(\mu) < 0$ for some value of μ, then the fixed point is a node. As μ changes, λ might increase (become less negative), going through zero, and then become positive. The node then changes to a repellor when $\lambda > 0$.

Es.Mappa Logistica 1D (non è un flusso ma il concetto è lo stesso!)

$$\boxed{x_{n+1} = Ax_n(1 - x_n)}$$



Biforcazioni in 1D

control parameter

Let us consider a specific example:

$$\boxed{\dot{x} = \mu - x^2} \qquad (3.17\text{-}3)$$

For μ positive, there are two fixed points: one at $x = +\sqrt{\mu}$, the other at $x = -\sqrt{\mu}$. For μ negative there are no fixed points (assuming, of course, that x is a real number). If we use Eq. (3.6-3), which defines the characteristic value for a fixed point, to find the characteristic value of the two fixed points (for $\mu > 0$), we see that the fixed point at $x = -\sqrt{\mu}$ is a repellor, while the fixed point at $x = +\sqrt{\mu}$ is a node.

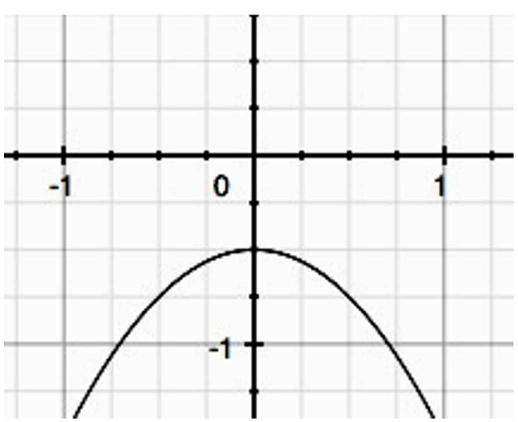
$$\lambda = \left.\frac{df(X)}{dX}\right|_{X=X_o}$$

$$\frac{df(X)}{dX} = -2x$$

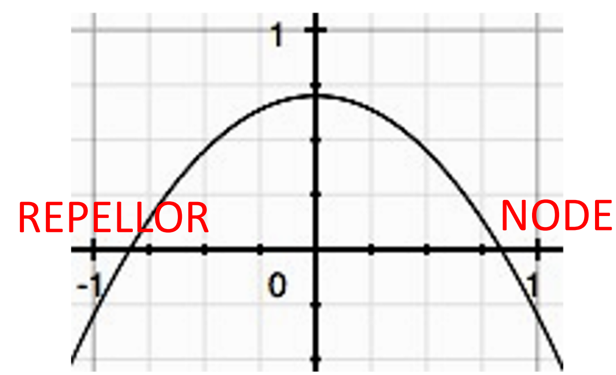
$$\lambda(+\sqrt{\mu}) < 0$$

$$\lambda(-\sqrt{\mu}) > 0$$

$\mu < 0$



$\mu > 0$



Biforcazioni in 1D

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Let us consider a specific example:

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If we start with $\mu < 0$ and let it increase, we find that a bifurcation takes place at $\mu = 0$. At that value of the parameter we have a saddle point, which then changes into a repellor-node pair as μ becomes positive. We say that we have a ***repellor-node bifurcation*** at $\mu = 0$.

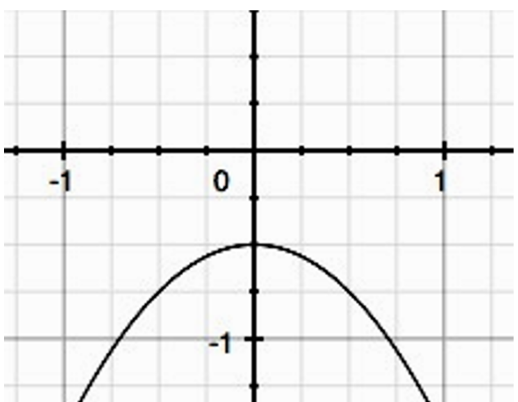
$$\lambda = \left.\frac{df(X)}{dX}\right|_{X=X_o}$$

$$\frac{df(X)}{dX} = -2x$$

$$\lambda(+\sqrt{\mu}) < 0$$

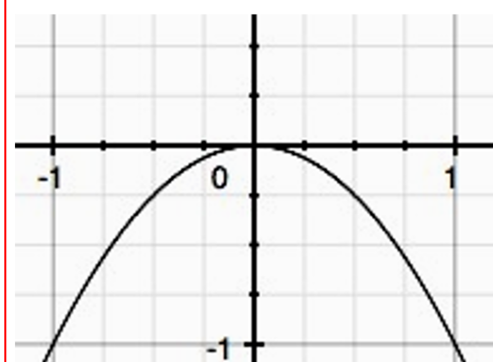
$$\lambda(-\sqrt{\mu}) > 0$$

$\mu < 0$

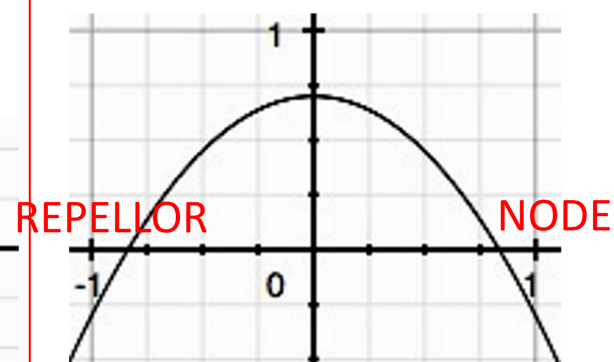


$\mu = 0$

BIFORCAZIONE!



$\mu > 0$



Biforcazioni in 1D

Let us consider a specific example:

control parameter

$$\boxed{\dot{x} = \mu - x^2} \qquad (3.17\text{-}3)$$

For μ positive, there are two fixed points: one at $x = +\sqrt{\mu}$, the other at $x = -\sqrt{\mu}$. For μ negative there are no fixed points (assuming, of course, that x is a real number). If we use Eq. (3.6-3), which defines the characteristic value for a fixed point, to find the characteristic value of the two fixed points (for $\mu > 0$), we see that the fixed point at $x = -\sqrt{\mu}$ is a repellor, while the fixed point at $x = +\sqrt{\mu}$ is a node.

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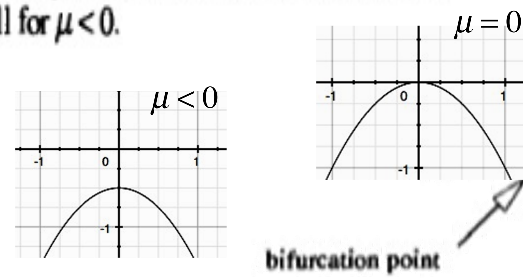
$$\lambda = \left.\frac{df(X)}{dX}\right|_{X=X_o}$$

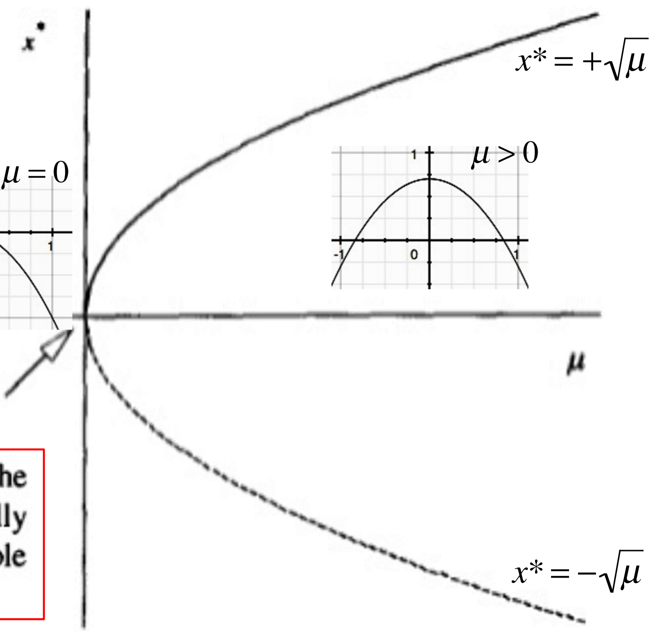
$$\frac{df(X)}{dX} = -2x$$

$$\lambda(+\sqrt{\mu}) < 0$$

$$\lambda(-\sqrt{\mu}) > 0$$

Fig. 3.14. The bifurcation diagram for the repellor-node (saddle-node) bifurcation. The solid line indicates the x value for the node as a function of the parameter value. The dashed line is for the repellor. Note that there is no fixed point at all for $\mu < 0$.





Nota: Note that at the repellor-node bifurcation point, the fixed point of the system is structurally unstable in the sense discussed in Section 3.6. Structurally unstable points are important because their existence indicates a possible bifurcation.

In the nonlinear dynamics literature, the bifurcation just described is usually called a ***saddle-node bifurcation***, ***tangent bifurcation***, or a ***fold bifurcation***. The origin of these names will become apparent when we see analogous bifurcations in higher-dimensional state spaces. For example, if we imagine the curves in Fig. 3.14 as being the cross section of a piece of paper extending into and out of the plane of the page, then the bifurcation point represents a "fold" in the piece of paper. Also, Fig. 3.5 shows how the function in question becomes tangent to the x axis at the bifurcation point.

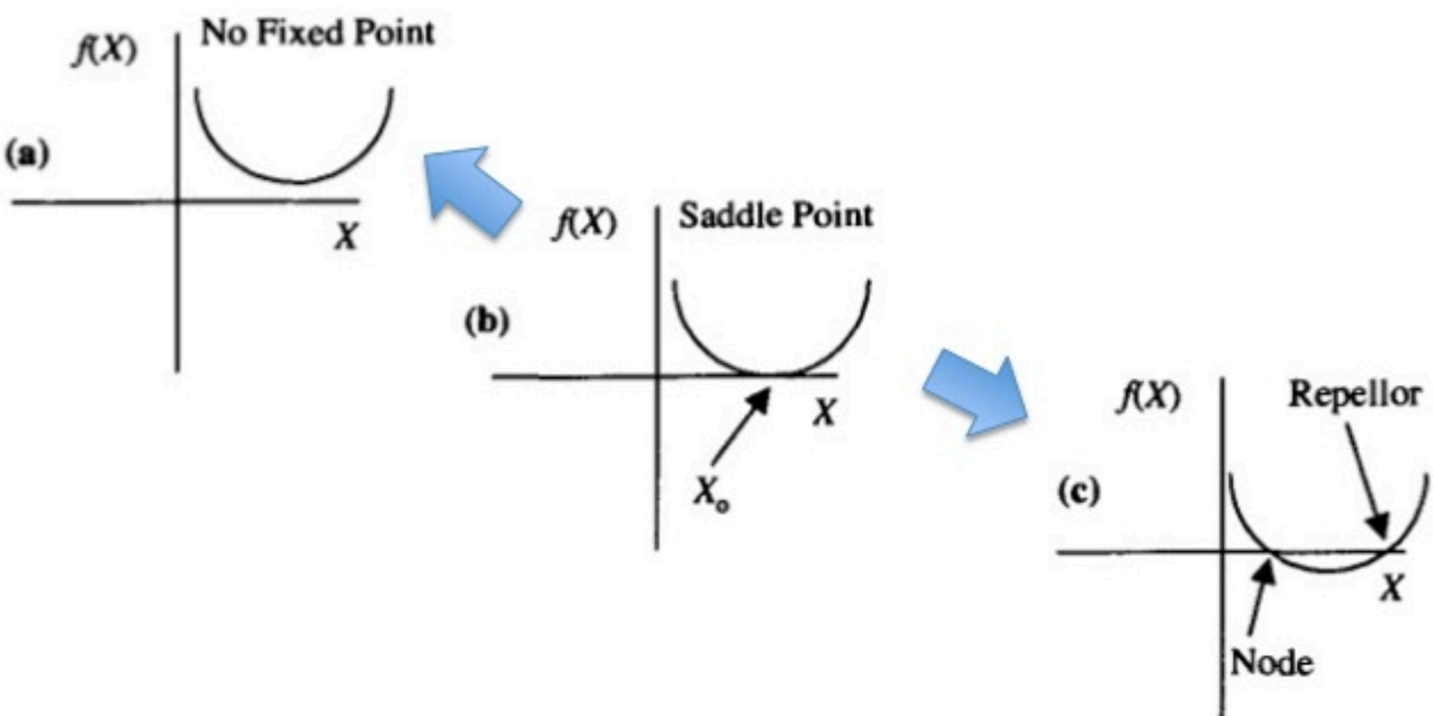
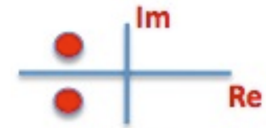


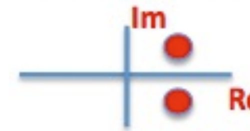
Fig. 3.5. In one-dimensional state spaces, a saddle point, the point X_o in (b), is structurally unstable. A small change in the function $f(X)$, for example pushing it up or down along the vertical axis, either removes the fixed point (a), or changes it into a node and a repellor (c).

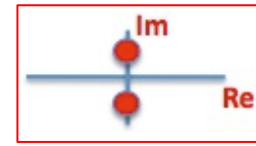
Biforcazioni in 2D

Limit Cycle Bifurcations

As we saw earlier, a fixed point in a two-dimensional state space may also have complex-valued characteristic values for which the trajectories have spiral-type behavior. A bifurcation occurs when the characteristic values move from the left-hand side of the complex plane to the right-hand side; that is, the bifurcation occurs when the real part of the characteristic value goes to 0.







We can also have limit cycle behavior in two-dimensional systems. The birth and death of a limit cycle are bifurcation events. The birth of a stable limit cycle is called a ***Hopf bifurcation*** (named after the mathematician E. Hopf). (Although this type of bifurcation was known and understood by Poincaré and later studied by the Russian mathematician A. D. Andronov in the 1930s, Hopf was the first to extend these ideas to higher-dimensional state spaces.) Since we can use a Poincaré section to study a limit cycle and since for a two-dimensional state space, the Poincaré section is just a line segment, the bifurcations of limit cycles can be studied by the same methods used for studying bifurcations of one-dimensional dynamical systems.

Es:
BRUSSELLATOR

A Hopf bifurcation can be modeled using the following normal form equations:

$$\dot{x}_1 = -x_2 + x_1\{\mu - (x_1^2 + x_2^2)\} \qquad (3.17\text{-}5a)$$

$$\dot{x}_2 = +x_1 + x_2\{\mu - (x_1^2 + x_2^2)\} \qquad (3.17\text{-}5b)$$

$$\dot{x}_1 = -x_2 + x_1\{\mu - (x_1^2 + x_2^2)\} \qquad (3.17\text{-}5a)$$

$$\dot{x}_2 = +x_1 + x_2\{\mu - (x_1^2 + x_2^2)\} \qquad (3.17\text{-}5b)$$

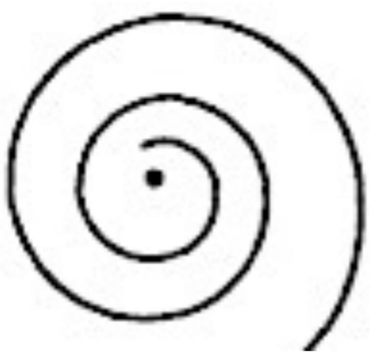
The geometric form of the trajectories is clearer if we change from (x_1, x_2) coordinates to polar coordinates (r,θ) defined in the following equations and illustrated in Fig. 3.18.

$$r = \sqrt{(x_1^2 + x_2^2)}$$
$$\tan\theta = \frac{x_2}{x_1} \qquad (3.17\text{-}6)$$

Using these polar coordinates, we write Eqs. (3.17-5) as

$$\dot{r} = r\{\mu - r^2\} \equiv f(r) \qquad (3.17\text{-}7a)$$

$$\dot{\theta} = 1 \rightarrow \theta(t) = \theta_o + t \qquad (3.17\text{-}7b)$$



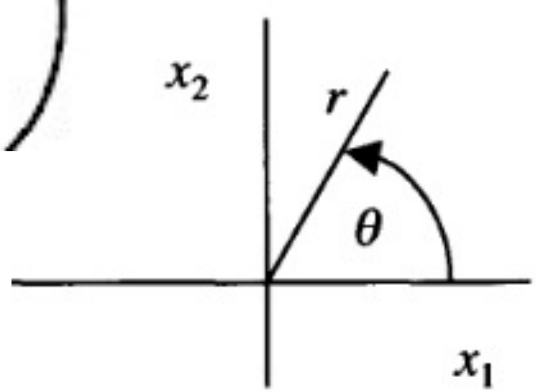


Fig. 3.18. The definition of polar coordinates. r is the length of the radius vector from the origin. θ is the angle between the radius vector and the positive x_1 axis.

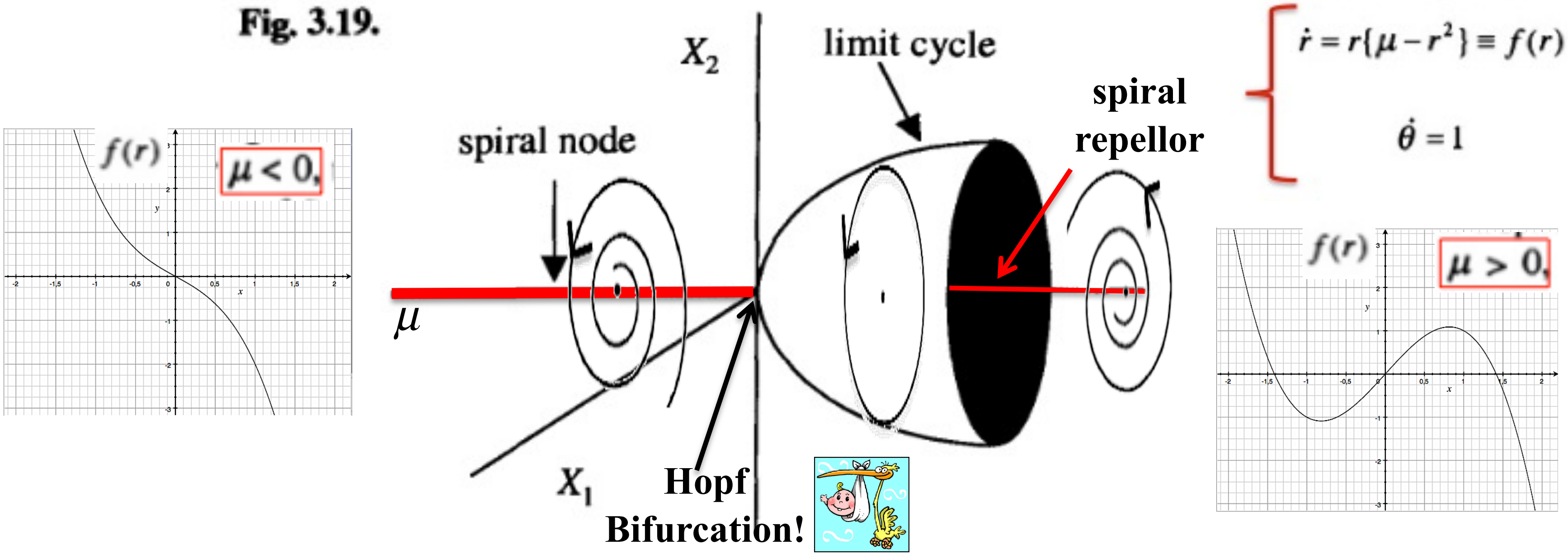
Now let us interpret the geometric nature of the trajectories that follow from Eqs. (3.17-7). The solution to Eq. (3.17-7b) is simply

$$\theta(t) = \theta_o + t \qquad (3.17\text{-}8)$$

that is, the angle continues to increase with time as the trajectory spirals around the origin. For $\boxed{\mu < 0}$, there is just one fixed point for r, namely $r = 0$. By evaluating the derivative of $f(r)$ with respect to r at $r = 0$, we see that the characteristic value is equal to μ. Thus, for $\mu < 0$, that derivative is negative, and the fixed point is stable. In fact, it is a spiral node.

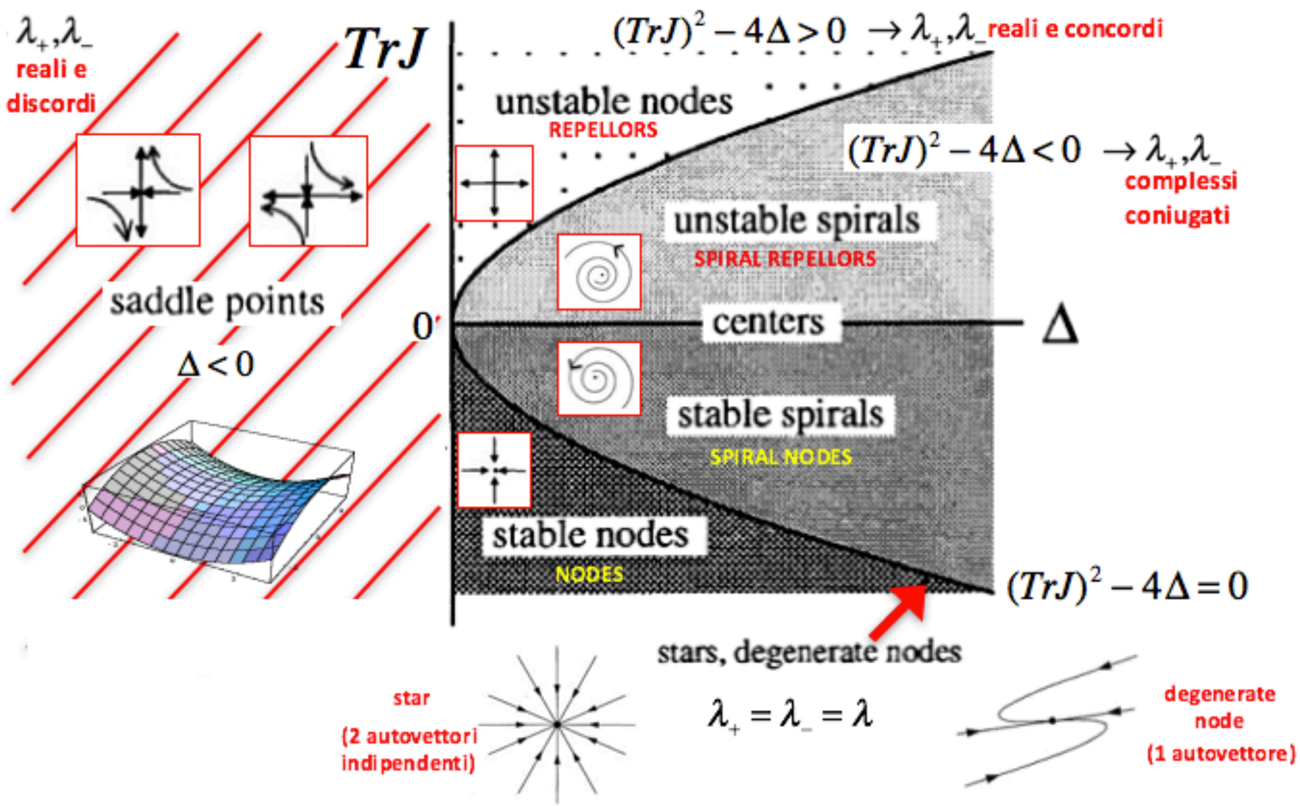
For $\boxed{\mu > 0}$, the fixed point at the origin is a spiral repellor; it is unstable; trajectories starting near the origin spiral away from it. There is, however, another fixed point for r, namely, $r = \sqrt{\mu}$. This fixed point for r corresponds to a limit cycle with a period of 2π [in the time units of Eqs. (3.17-7)]. We say that the limit cycle is born at the bifurcation value $\mu = 0$. Fig. 3.19 shows the bifurcation diagram for the Hopf bifurcation.

Fig. 3.19.

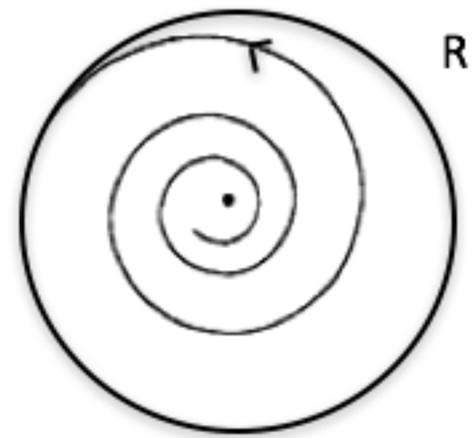


3.18 Summary

In this chapter we have developed much of the mathematical machinery needed to discuss the behavior of dynamical systems. We have seen that fixed points and their characteristic values (determined by derivatives of the functions describing the dynamics of the system) are crucial for understanding the dynamics. We have also seen that the dimensionality of the state space plays a major role in determining the kinds of trajectories that can occur for bounded systems.



Il Teorema di Poincaré-Bendixson



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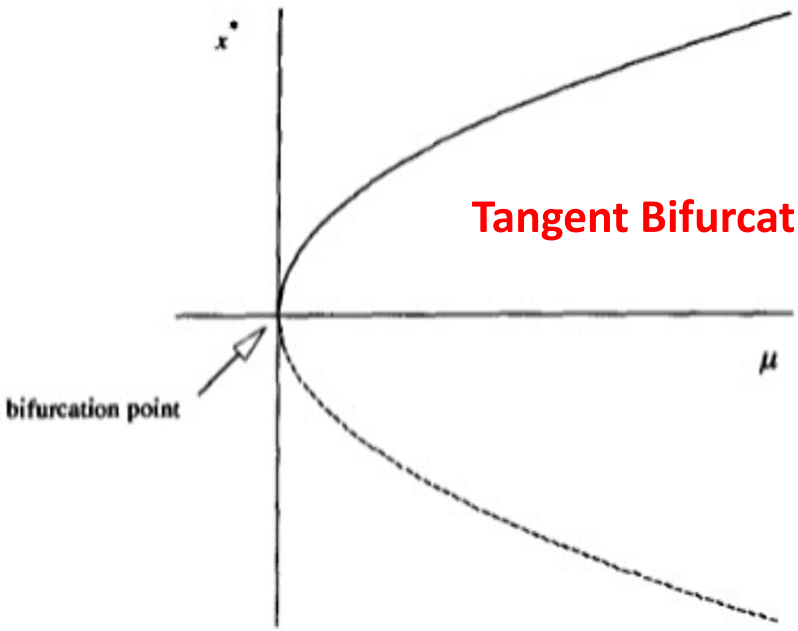
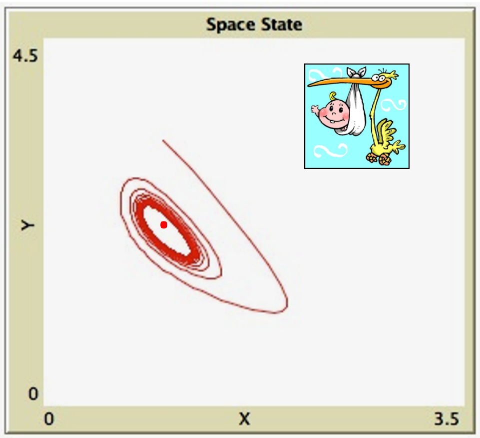
Moreover, as the control parameters of a system change, the character of fixed points and the nature of trajectories near them can change dramatically at bifurcation points. Bifurcation diagrams are used to describe the change in behavior near bifurcation points. We again saw that the dimensionality of the state space limits the kinds of bifurcations that can commonly occur.



Tangent Bifurcation (1D)

Hopf Bifurcation (2D)



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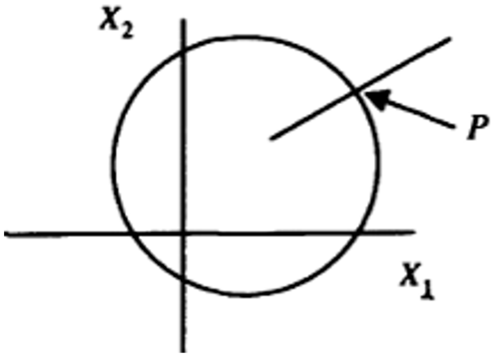
In state spaces with two or more dimensions, limit cycles, describing periodic behavior, can appear. The stability of a limit cycle can be discussed by means of a Poincaré section and the characteristic multiplier determined by the derivative of the corresponding Poincaré map function. A limit cycle may be born via a Hopf bifurcation.



Sezione di Poincaré

3.18 Summary

In this chapter we have developed much of the mathematical machinery needed to discuss the behavior of dynamical systems. We have seen that fixed points and their characteristic values (determined by derivatives of the functions describing the dynamics of the system) are crucial for understanding the dynamics. We have also seen that the dimensionality of the state space plays a major role in determining the kinds of trajectories that can occur for bounded systems.

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In state spaces with two or more dimensions, limit cycles, describing periodic behavior, can appear. The stability of a limit cycle can be discussed by means of a Poincaré section and the characteristic multiplier determined by the derivative of the corresponding Poincaré map function. A limit cycle may be born via a Hopf bifurcation.

In a two-dimensional state space the possible types of bifurcations are also limited. As explained in Appendix B, the saddle-node bifurcation and the Hopf bifurcation are the most "common" two-dimensional bifurcations for models with one control parameter. As we shall see in the next chapter, once we move to a state space with three or more dimensions the number of common bifurcations increases tremendously.

Ex.1 ROMEO E GIULIETTA

Il libro di Strogatz suggerisce di studiare, come esercizio, un **sistema dinamico lineare** a due dimensioni che descrive, al variare dei parametri, la **variazione temporale dell'amore o dell'odio tra due partner coinvolti in una relazione romantica**.

Definiamo x(t) come l'amore (o l'odio nel caso in cui sia negativo) di Romeo nei confronti di Giulietta al tempo "t" e y(t) l'amore (o l'odio) di Giulietta nei confronti di Romeo. Così abbiamo le seguenti **due equazioni differenziali del primo ordine:**

Romeo $\dot{x} = ax + by$

Giulietta $\dot{y} = cx + dy$

I **parametri "a" e "b" stabiliscono il comportamento di Romeo mentre "c" e "d" quello di Giulietta**; "a" descrive l'attrazione di Romeo causata dai suoi stessi sentimenti, mentre "b" l'attrazione causata dai sentimenti di Giulietta, e lo stesso vale per Giulietta. Ad esempio, **Romeo** può mostrare 4 comportamenti diversi in base al segno dei parametri "a" e "b":

Appassionato: a>0; b>0 (Romeo è spinto dai suoi stessi sentimenti così come da quelli di Giulietta)

Narcisistico: a>0; b<0 (Romeo è spinto ancora dai suoi sentimenti ma indietreggia a causa dei sentimenti di Giulietta)

Amanti prudenti: a<0; b>0 (Romeo si tira indietro sui suoi stessi sentimenti ma è incoraggiato da Giulietta)

Eremita: a<0; b<0 (Romeo si tira indietro sui suoi stessi sentimenti così come da Giulietta)

Esercizio:

Esplorare il modello sia analiticamente che con l'aiuto di NetLogo in corrispondenza di diversi valori dei parametri

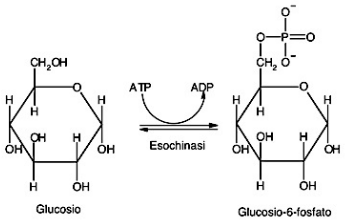
Ex.2 LA GLICOLISI

In the fundamental biochemical process called ***glycolysis***, living cells obtain energy by breaking down sugar. In intact yeast cells as well as in yeast or muscle extracts, glycolysis can proceed in an *oscillatory* fashion, with the concentrations of various intermediates waxing and waning with a period of several minutes. For reviews, see Chance et al. (1973) or Goldbeter (1980).

A simple model of these oscillations has been proposed by Sel'kov (1968). In dimensionless form, the equations are

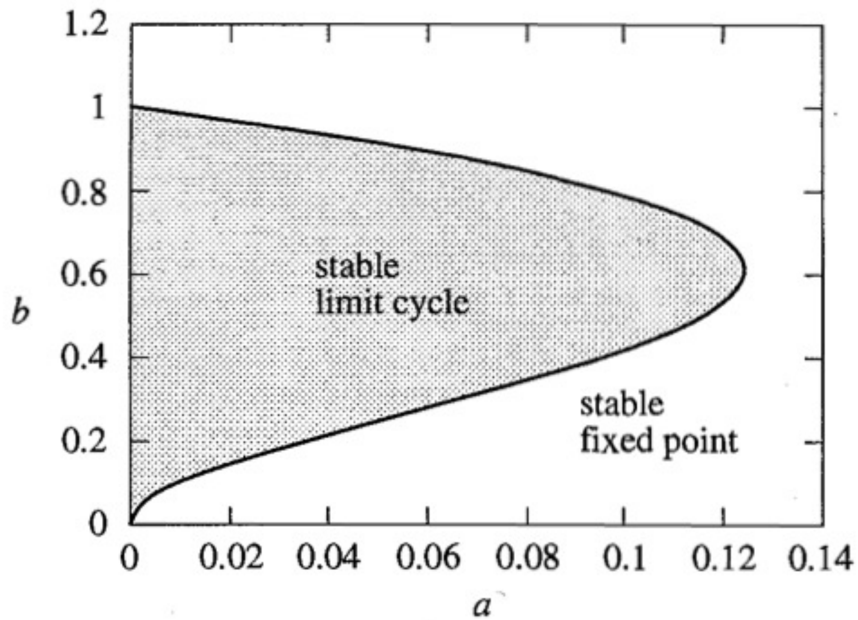
$$\dot{x} = -x + ay + x^2 y$$
$$\dot{y} = b - ay - x^2 y$$

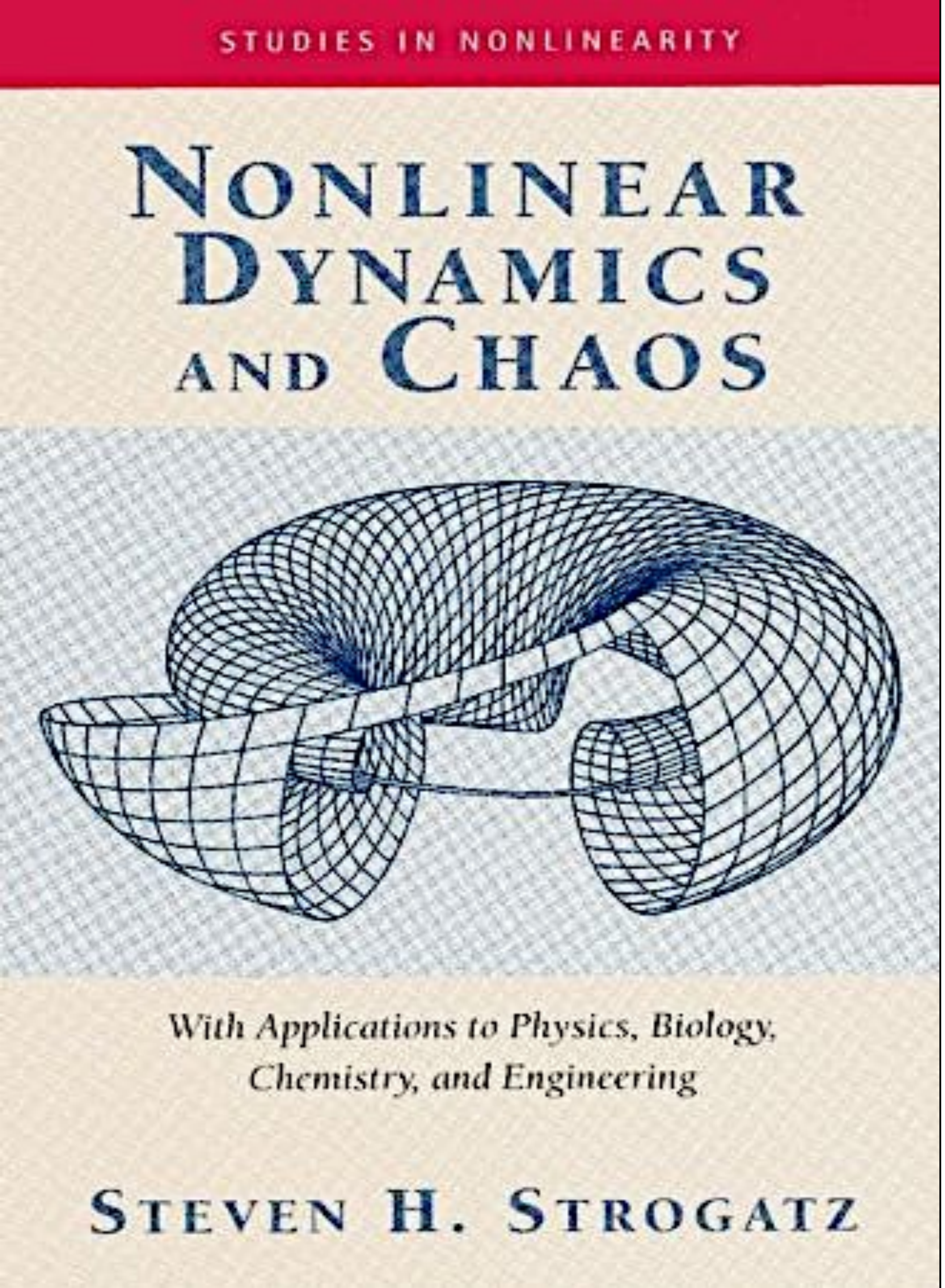
Valori tipici: a=0.08, b=0.6



where x and y are the concentrations of ADP (adenosine diphosphate) and F6P (fructose-6-phosphate), and $a, b > 0$ are kinetic parameters.

<u>Esercizio</u>:
Esplorare il modello sia analiticamente che con l'aiuto di NetLogo





Sullo Strogatz potete trovare molti altri spunti per lo studio analitico e numerico di sistemi dinamici a 2 dimensioni…