Classificazione dei Sistemi Dinamici

Sistemi dinamici continui (Flussi)

$$\dot{X} = f(X)$$

Flussi Dissipativi

Attrattori

1D Punto fisso

2D Ciclo Limite

3D Caotici

Flussi Hamiltoniani

Orbite

Periodiche

Quasi Periodiche

Caotiche

Sistemi dinamici discreti (Mappe)

$$x_{n+1} = Ax_n(1-x_n) \equiv f_A(x)$$

Mappe Dissipative

Attrattori

Punto fisso

Ciclo Limite

Caotici

Mappe Conservative (area-preserving)

Orbite

Periodiche

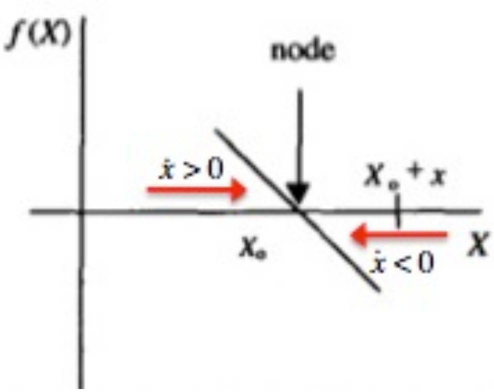
Quasi Periodiche

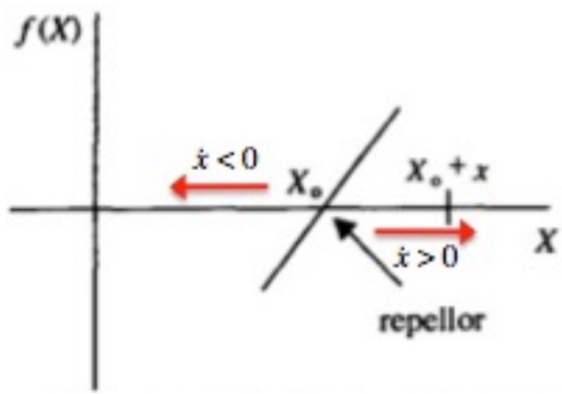
Caotiche

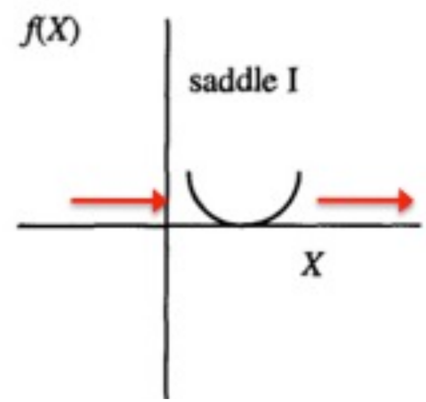
Flussi dissipativi in una dimensione

$$\frac{1}{L}\frac{dL}{dt} = \frac{1}{L}[f(X_B) - f(X_A)] = \frac{df(X)}{dX} < 0$$

fixed points (dim.0)





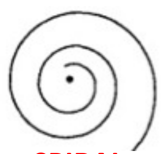


Flussi dissipativi
in
due dimensioni

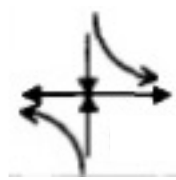
$$\frac{1}{A}\frac{dA}{dt} = \frac{\partial f_1}{\partial X_1} + \frac{\partial f_2}{\partial X_2} \quad < 0$$

fixed points (dim.0)

NODE

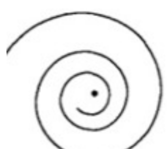


SPIRAL NODE

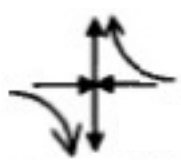


SADDLE POINTS

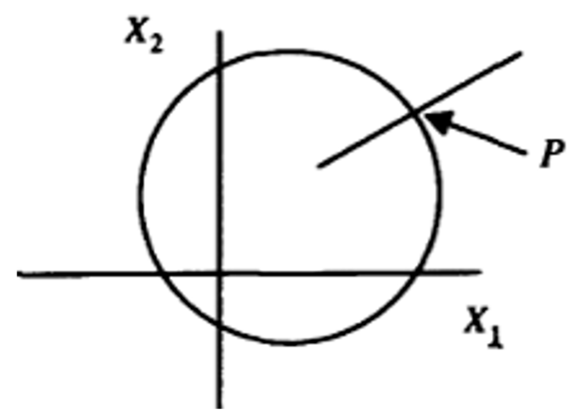
REPELLOR



SPIRAL REPELLOR

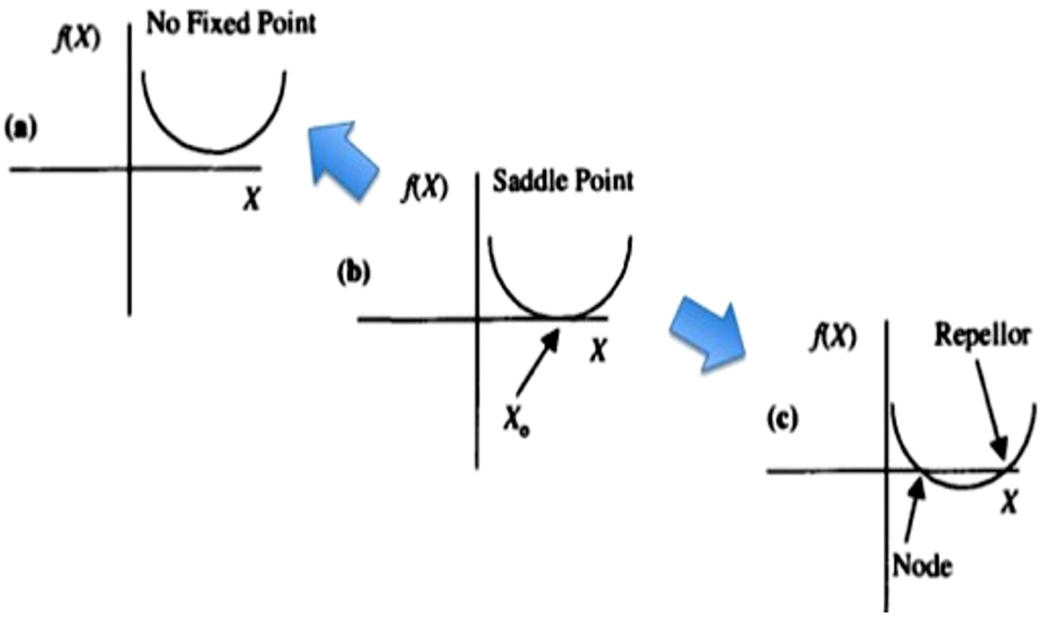


limit cycles (dim.1)

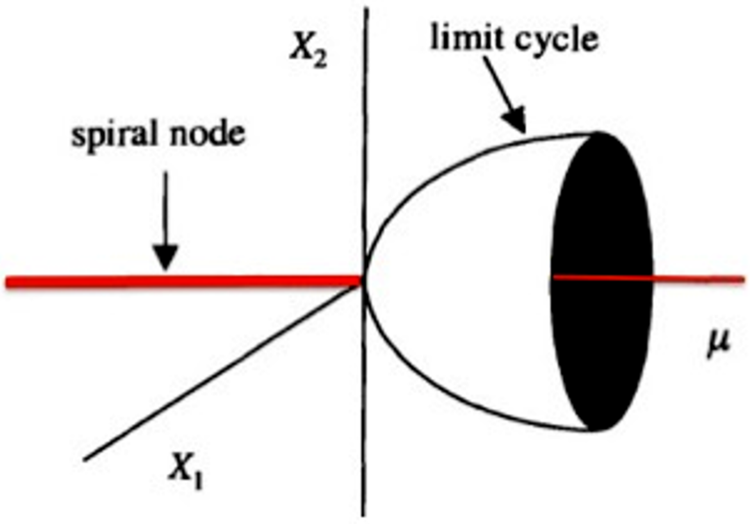


Biforcazioni

1D-2D: Repellor-Node (or Saddle-Node or Tangent) Bifurcation

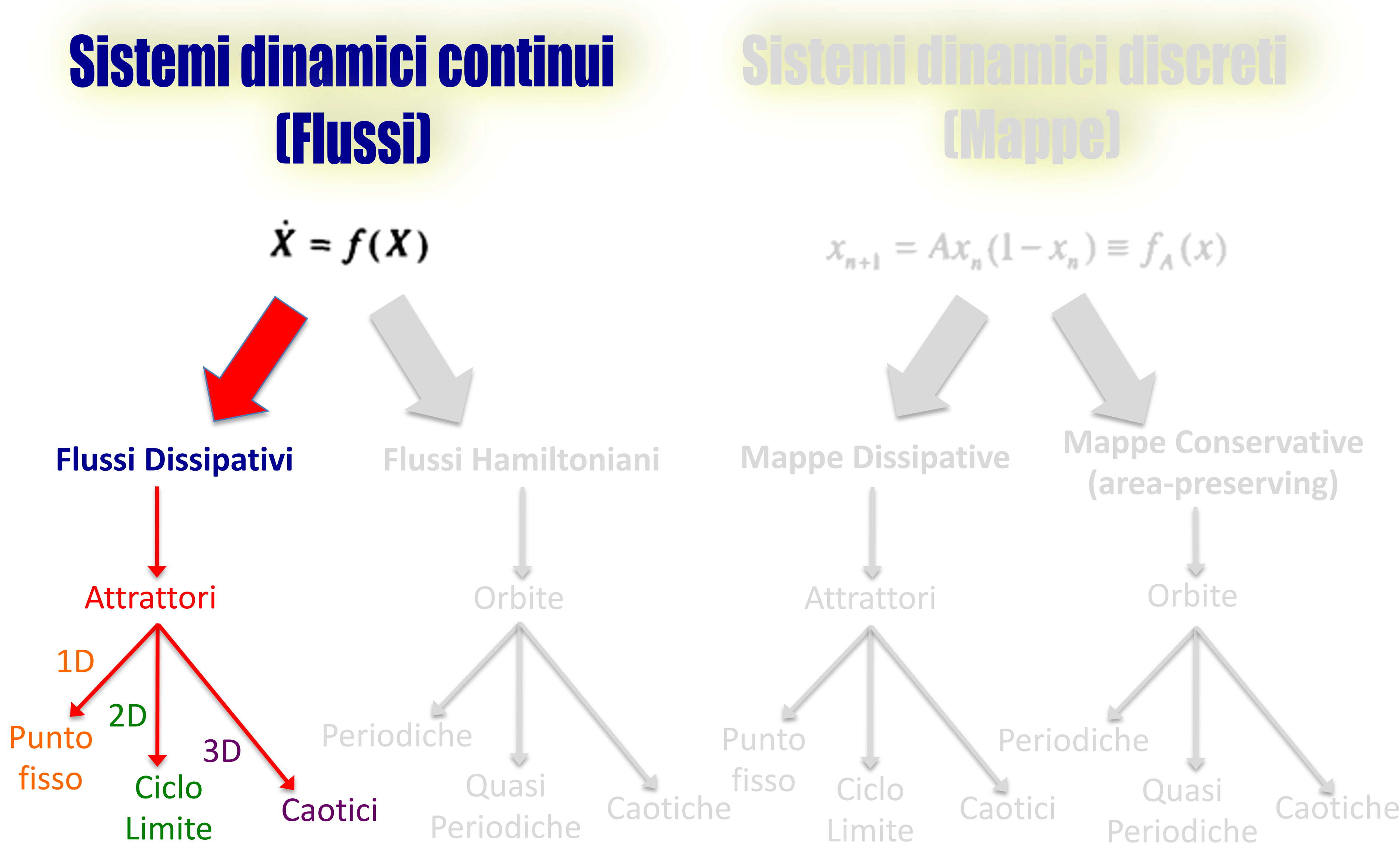


2D: Hopf Bifurcation



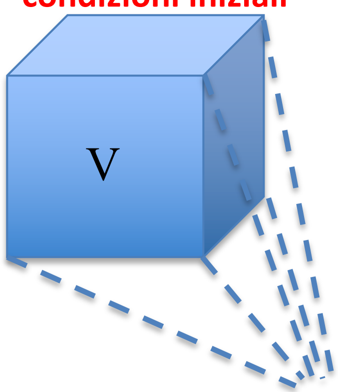
3D: . . . (Rotte verso il Caos...)

Classificazione dei Sistemi Dinamici



Flussi dissipativi in tre dimensioni

Cluster di condizioni iniziali



$$\frac{1}{V}\frac{dV}{dt} = \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i} \equiv div(f) < 0$$

ATTRATTORI

- fixed points (dim.0)
- limit cycles (dim.1)
- quasiperiodic attractors (dim.2)
- chaotic attractors (fractal dimension between 2 and 3)

4

Three-Dimensional State Space and Chaos

4.1 Overview

In the previous chapter, we introduced some of the standard methods for analyzing dynamical systems described by systems of ordinary differential equations, but we limited the discussion to state spaces with one or two dimensions. We are now ready to take the important step to three dimensions. This is a crucial step, not because we live in a three-dimensional world (remember that we are talking about state space, not physical space), but because in three dimensions dynamical systems can behave in ways that are not possible in one or two dimensions. Foremost among these new possibilities is chaos.

First we will give a hand-waving argument (we could call it heuristic if we wanted to sound more sophisticated) that shows why chaotic behavior may occur in three dimensions. We will then discuss, in parallel with the treatment of the previous chapter, a classification of the types of fixed points that occur in three dimensions. However, we gradually wean ourselves from the standard analytic techniques and begin to rely more and more on graphic and geometrical (topological) arguments. This change reflects the flavor of current developments in dynamical systems theory. In fact, the main goal of this chapter is to develop geometrical pictures of trajectories, attractors, and bifurcations in three-dimensional state spaces.

4.2 Heuristics

We will describe, in a rather loose way, why three (or more) state space dimensions are needed to have chaotic behavior. First, we should remind ourselves that we are dealing with dissipative systems whose trajectories eventually approach an attractor. For the moment we are concerned only with the trajectories that have settled into the attracting region of state space. When we write about the divergence of nearby trajectories, we are concerned with the behavior of trajectories within the attracting region of state space.

In a somewhat different context we will need to consider sensitive dependence on initial conditions. Initial conditions that are not, in general, part of an attractor can lead to very different long-term behaviors on different attractors. Those behaviors, determined by the nature of the attractor (or attractors), might be time-independent or periodic or chaotic.

As we saw in Chapter 1, chaotic behavior is characterized by the divergence of nearby trajectories in state space. As a function of time, the "separation" (suitably defined) between two nearby trajectories increases exponentially, at least for short times. The last restriction is necessary because we are concerned with systems whose trajectories stay within some bounded region of state space. The system does not "blow up." There are three requirements for chaotic behavior in such a situation:

1. no intersection of different trajectories;
2. bounded trajectories;
3. exponential divergence of nearby trajectories.

These conditions cannot be satisfied simultaneously in one- or two-dimensional state spaces. You should convince yourself that this is true by sketching some trajectories in a two-dimensional state space on a sheet of paper. However, in three dimensions, initially nearby trajectories can continue to diverge by wrapping over and under each other. Obviously sketching three-dimensional trajectories is more difficult. You might try using some relatively stiff wire to form some trajectories in three dimensions to show that all three requirements for chaotic behavior can be met. You should quickly discover that these requirements lead to trajectories that initially diverge, then curve back through the state space, forming in the process an intricate layered structure. Figure 4.1 is a sketch of diverging trajectories in a three-dimensional state space.

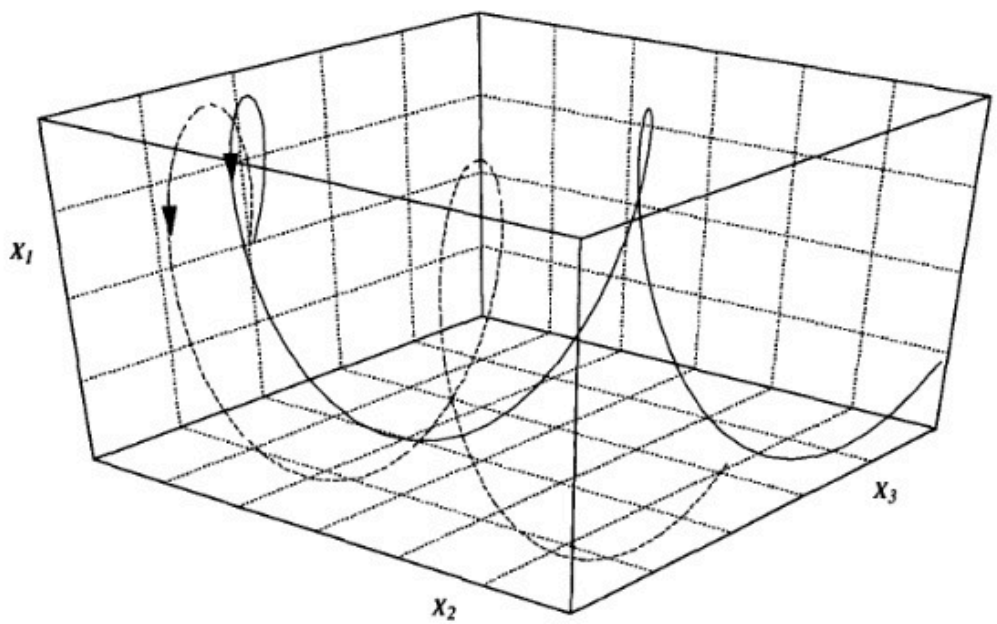


Fig. 4.1. A sketch of trajectories in a three-dimensional state space. Notice how two nearby trajectories can continue to behave quite differently from each other yet remain bounded by weaving in and out and over and under each other.

The crucial feature of state space with three or more dimensions that permits chaotic behavior is the ability of trajectories to remain within some bounded region by intertwining and wrapping around each other (without intersecting!) and without repeating themselves exactly. Clearly the geometry associated with such trajectories is going to be strange. In fact, such attractors are now called ***strange attractors***. In Chapter 9, we will give a more precise definition of a strange attractor in terms of the notion of fractal dimension. If the behavior on the attractor is chaotic, that is, if the trajectories on the attractor display exponential divergence of nearby trajectories (on the average), then we say the attractor is chaotic. Many authors use the terms *strange attractor* and ***chaotic attractor*** interchangeably, but in principle they are distinct [GOP84].

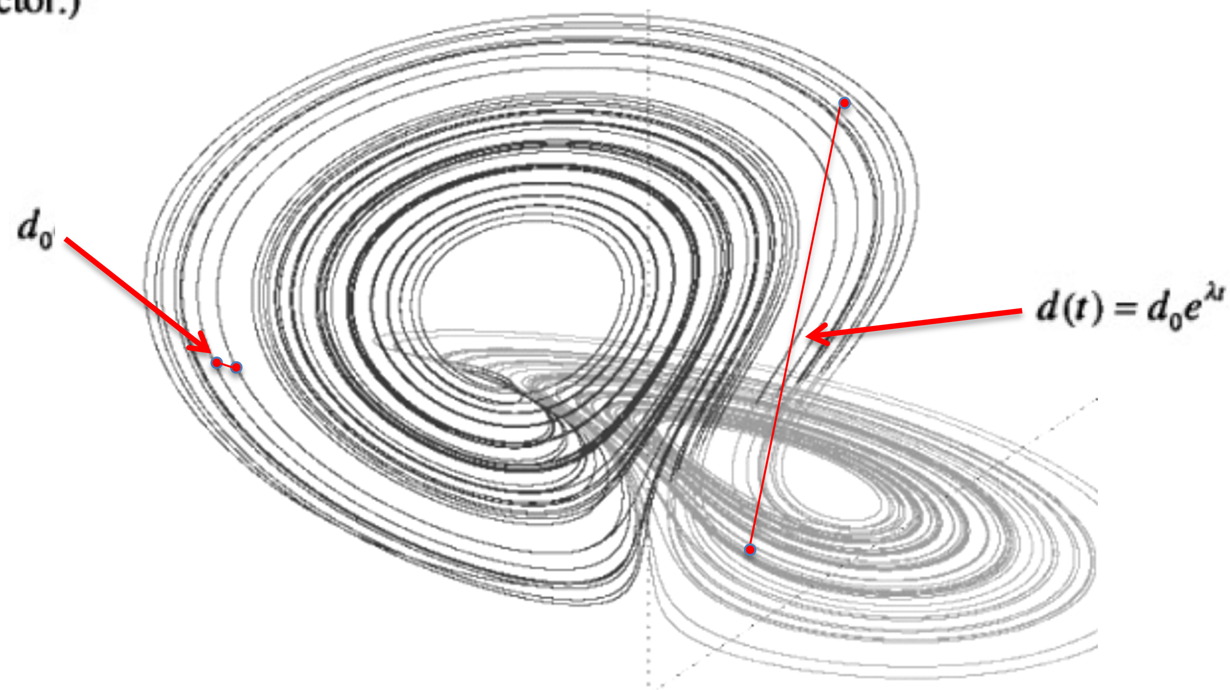
The notion of exponential divergence of nearby trajectories is made formal by introducing the ***Lyapunov exponent***. If two nearby trajectories on a chaotic attractor start off with a separation d_0 at time $t = 0$, then the trajectories diverge so that their separation at time t, denoted by $d(t)$, satisfies the expression

$$d(t) = d_0 e^{\lambda t} \qquad (4.2\text{-}1)$$

The parameter λ in Eq. (4.2-1) is called the Lyapunov exponent for the trajectories. If λ is positive, then we say the behavior is chaotic. (Section 4.13 takes up the question of Lyapunov exponents in more detail.) From this definition of chaotic behavior, we see that chaos is a property of a collection of trajectories.

Chaos, however, also appears in the behavior of a single trajectory. As the trajectory wanders through the (chaotic) attractor in state space, it will eventually return near some point it previously visited. (Of course, it cannot return exactly to that point. If it did, then the trajectory would be periodic.) If the trajectories exhibit exponential divergence, then the trajectory on its second visit to a particular neighborhood will have subsequent behavior, quite different from its behavior on the first visit. Thus, the impression of the time record of this behavior will be one of nonreproducibility, nonperiodicity, in short, of chaos.

The point of these remarks is to remind us that our notions of sensitive dependence on initial conditions and divergence of nearby trajectories are meaningful and useful only for those systems that are bounded and have attractors in the sense defined in Chapter 3. (In Chapter 8, we shall see how to generalize these ideas to bounded Hamiltonian systems for which there is no dissipation and no attractor.)



4.4 Three-Dimensional Dynamical Systems

We will now introduce some of the formalism for the description of a dynamical system with three state variables. We call a dynamical system three-dimensional if it has three independent dynamical variables, the values of which at a given instant of time uniquely specify the state of the system. We assume that we can write the time-evolution equations for the system in the form of the standard set of first-order ordinary differential equations. (Dynamical systems modeled by iterated map functions will be discussed in Chapter 5.) Here we will use x with a subscript 1, 2, or 3 to identify the variables. This formalism can then easily be generalized to any number of dimensions simply by increasing the numerical range of the subscripts. The differential equations take the form

$$\dot{X} = p(Y - X)$$
$$\dot{Y} = -XZ + rX - Y$$
$$\dot{Z} = XY - bZ$$

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, x_3) \\ \dot{x}_2 &= f_2(x_1, x_2, x_3) \\ \dot{x}_3 &= f_3(x_1, x_2, x_3) \end{aligned} \tag{4.4-1}$$

The Lorenz model equations of Chapter 1 are of this form. Note that the three functions f_1, f_2, and f_3 do not involve time explicitly; again, we say that the system is autonomous.

As an aside, we note that some authors like to use a symbolic "vector" form to write the system of equations:

$$\dot{\vec{x}} = \vec{f}(\vec{x}) \tag{4.4-2}$$

Here $\vec{x}$ stands for the three symbols $x_1, x_2 x_3$, and $\vec{f}$ stands for the three functions on the right-hand side of Eqs. (4.4-1).

The differential equations describing two-dimensional systems subject to a time-dependent "force" (and hence nonautonomous) can also be written in the form of Eq. (4.4-1) by making use of the "trick" introduced in Chapter 3: Suppose that the two-dimensional system is described by equations of the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, t) \\ \dot{x}_2 &= f_2(x_1, x_2, t) \end{aligned} \tag{4.4-3}$$

The trick is to introduce a third variable, $x_3 = t$. The three "autonomous" equations then become

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, x_3) \\ \dot{x}_2 &= f_2(x_1, x_2, x_3) \\ \dot{x}_3 &= 1 \end{aligned} \tag{4.4-4}$$

which are of the same form as Eq. (4.4-1). As we shall see, this trick is particularly useful when the time-dependent term is periodic in time.

Exercise 4.4-1. The "forced" van der Pol equation is used to describe an electronic triode tube circuit subject to a periodic electrical signal. The equation for $q(t)$, the charge oscillating in the circuit, can be put in the form

$$\frac{d^2q}{dt^2} + \gamma(q)\frac{dq}{dt} + q(t) = g \sin \omega t$$

Use the trick introduced earlier to write this equation in the standard form of Eq. (4.4-1).

4.5 Fixed Points in Three Dimensions (dim = 0)

The fixed points of the system of Eqs. (4.4-1) are found, of course, by setting the three time derivatives equal to 0. [Two-dimensional forced systems, even if written in the three-dimensional form (4.4-4), do not have any fixed points because, as the last of Eqs. (4.4-4) shows, we never have $\dot{x}_3 = t = 0$. Thus, we will need other techniques to deal with them.] The nature of each of the fixed points is determined by the three characteristic values of the Jacobian matrix of partial derivatives evaluated at the fixed point in question. The Jacobian matrix is

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{pmatrix} \tag{4.5-1}$$

In finding the characteristic values of this matrix, we will generally have a cubic equation, whose roots will be the three characteristic values labeled $\lambda_1, \lambda_2, \lambda_3$.

Some mathematical details: The standard theory of cubic equations tells us that a cubic equation of the form

$$\boxed{\lambda^3 + p\lambda^2 + q\lambda + r = 0} \tag{4.5-2}$$

can be changed to the "standard" form

$$x^3 + ax + b = 0 \tag{4.5-3}$$

by the use of the substitutions

$$\begin{aligned} x &= \lambda + p/3 \\ a &= \frac{1}{3}(3q - p^2) \\ b &= \frac{1}{27}(2p^3 - 9qp + 27r) \end{aligned} \tag{4.5-4}$$

If we now introduce

$$\boxed{\begin{aligned} s &= \left(\frac{b^2}{4} + \frac{a^3}{27}\right) \\ A &= (-b/2 + \sqrt{s})^{\frac{1}{3}} \\ B &= (-b/2 - \sqrt{s})^{\frac{1}{3}} \end{aligned}} \tag{4.5-5}$$

the three roots of the x equation can be written as

$$\begin{aligned}
\lambda_1 &= A+B \\
\lambda_2 &= -\left(\frac{A+B}{2}\right)+\left(\frac{A-B}{2}\right)\sqrt{-3} \\
\lambda_3 &= -\left(\frac{A+B}{2}\right)-\left(\frac{A-B}{2}\right)\sqrt{-3}
\end{aligned} \tag{4.5-6}$$

from which the characteristic values for the matrix can be found by working back through the set of substitutions. Most readers will be greatly relieved to know that we will not make explicit use of these equations. But it is important to know the <u>form</u> of the solutions.

WolframAlpha computational knowledge engine

Enter what you want to calculate or know about:

Examples Random

the three roots of the x equation can be written as

$$\begin{aligned} \lambda_1 &= A+B \\ \lambda_2 &= -\left(\frac{A+B}{2}\right)+\left(\frac{A-B}{2}\right)\sqrt{-3} \\ \lambda_3 &= -\left(\frac{A+B}{2}\right)-\left(\frac{A-B}{2}\right)\sqrt{-3} \end{aligned} \tag{4.5-6}$$

from which the characteristic values for the matrix can be found by working back through the set of substitutions. Most readers will be greatly relieved to know that we will not make explicit use of these equations. But it is important to know the <u>form</u> of the solutions.

There are three cases to consider:

"standard" form

$x^3 + ax + b = 0$

$$s = \left(\frac{b^2}{4} + \frac{a^3}{27}\right)$$

1. The three characteristic values are real and unequal ($s < 0$).
2. The three characteristic values are real and at least two are equal ($s = 0$).
3. There is one real characteristic value and two complex conjugate values ($s > 0$).

Case 2 is just a borderline case and need not be treated separately.

Punti Fissi in uno Spazio degli Stati a Tre Dimensioni

The four basic types of fixed points for a three-dimensional state space are:

1. *Node*. All the characteristic values are real and negative. All trajectories in the neighborhood of the node are attracted toward the fixed point without looping around the fixed point.

   1s. ***Spiral Node***. All the characteristic values have negative real parts but two of them have nonzero imaginary parts (and in fact form a complex conjugate pair). The trajectories spiral around the node on a "surface" as they approach the node.

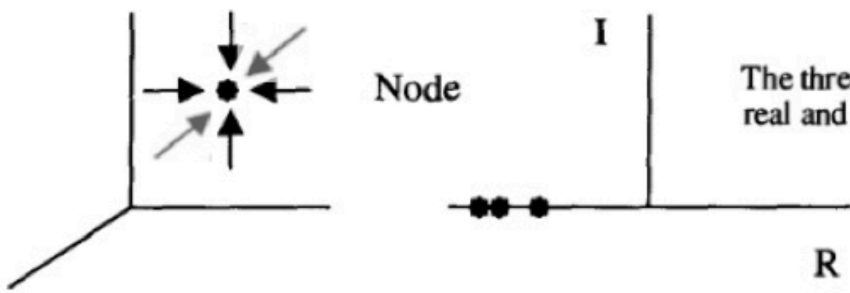
Equazione caratteristica:

$$\lambda^3 + p\lambda^2 + q\lambda + r = 0$$

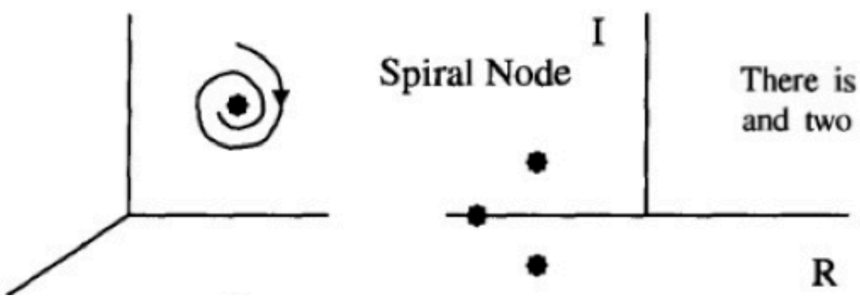
"standard" form

$$x^3 + ax + b = 0$$

$$s = \left(\frac{b^2}{4} + \frac{a^3}{27}\right)$$



The three characteristic values are real and unequal ($s < 0$).



There is one real characteristic value and two complex conjugate values ($s > 0$).

Punti Fissi in uno Spazio degli Stati a Tre Dimensioni

The four basic types of fixed points for a three-dimensional state space are:

2. ***Repellor***. All the characteristic values are real and positive. All trajectories in the neighborhood of the repellor diverge from the repellor.

   2s. ***Spiral Repellor***. All the characteristic values have positive real parts, but two of them have nonzero imaginary parts (and in fact form a complex conjugate pair). Trajectories spiral around the repellor (on a "surface") as they are repelled from the fixed point.

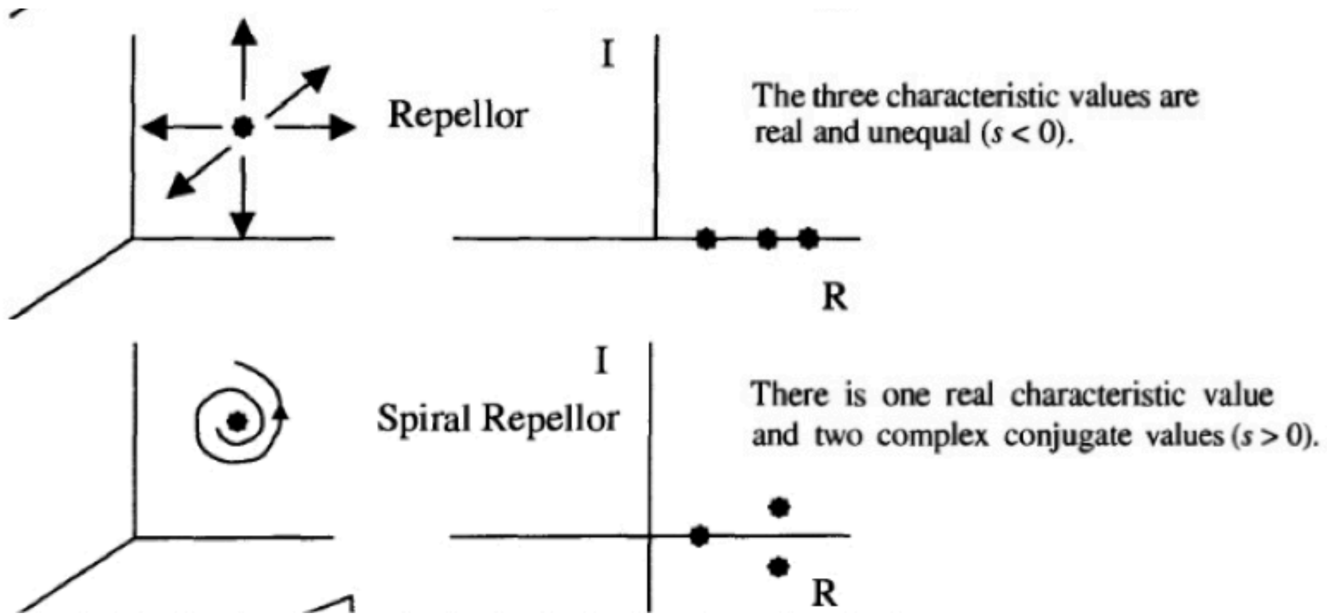
Equazione caratteristica:

$$\lambda^3 + p\lambda^2 + q\lambda + r = 0$$

"standard" form

$$x^3 + ax + b = 0$$

$$s = \left(\frac{b^2}{4} + \frac{a^3}{27}\right)$$



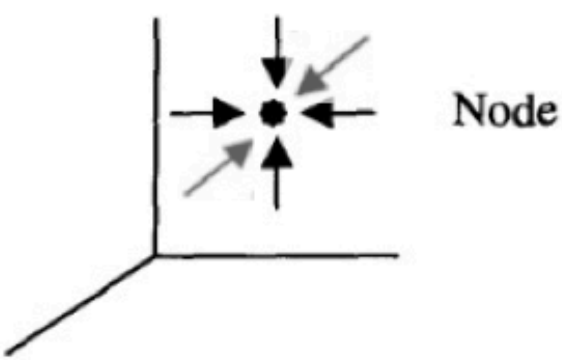
Punti Fissi in uno Spazio degli Stati a Tre Dimensioni

For state spaces with three or more dimensions, it is common to specify the so-called *index* of a fixed point.

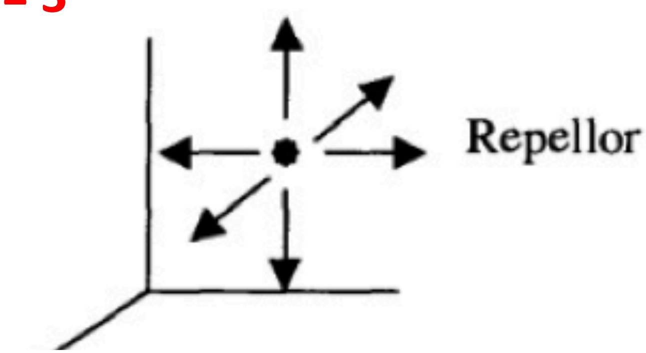
> The ***index*** of a fixed point is defined to be the number of characteristic values of that fixed point whose real parts are positive.

In more geometric terms, the index is equal to the spatial dimension of the out-set of that fixed point. For a node (which does not have an out-set), the index is equal to 0. For a repellor, the index is equal to 3 for a three-dimensional state space. A saddle point can have either an index of 1, if the out-set is a curve, or an index of 2, if the out-set is a surface as shown in Fig. 4.3.

Index = 0



Index = 3



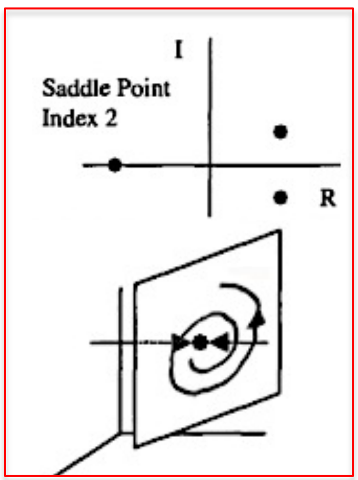
Punti Fissi in uno Spazio degli Stati a Tre Dimensioni

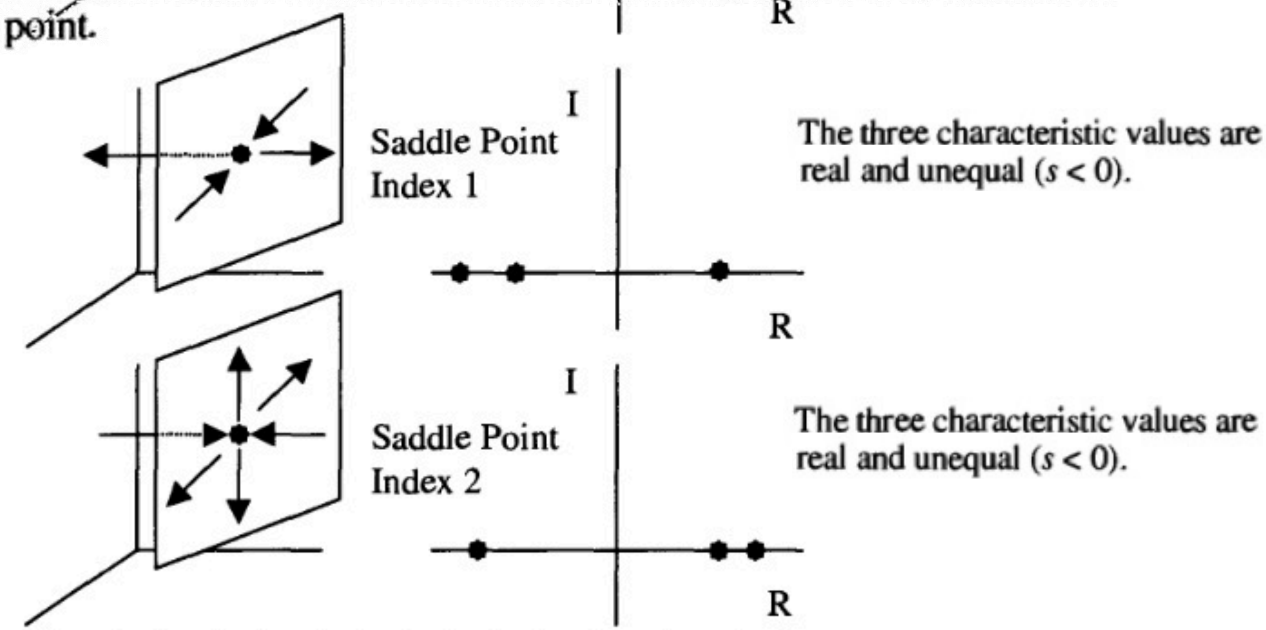
3. ***Saddle point — index–1.*** All characteristic values are real. One is positive and two are negative. Trajectories approach the saddle point on a surface (the in-set) and diverge along a curve (the out-set).

   3s. ***Spiral Saddle Point — index–1.*** The two characteristic values with negative real parts form a complex conjugate pair. Trajectories spiral around the saddle point as they approach on the in-set surface.

4. ***Saddle point — index–2.*** All characteristic values are real. Two are positive and one is negative. Trajectories approach the saddle point on a curve (the in-set) and diverge from the saddle point on a surface (the out-set).

   4s. ***Spiral Saddle Point — index–2.*** The two characteristic values with positive real parts form a complex conjugate pair. Trajectories spiral around the saddle point on a surface (the out-set) as they diverge from the saddle point.





4.6 Limit Cycles and Poincaré Sections (dim = 1)

As we saw in Chapter 3, dynamical systems in two (and higher) dimensions can also settle into long-term behavior associated with repetitive, periodic limit cycles. We also learned that the Poincaré section technique can be used to reduce the dimensionality of the description of these limit cycles and to make their analysis simpler.

First, we focus on the construction of a Poincaré section for the system. For a three-dimensional state space, the Poincaré section is generated by choosing a ***Poincaré plane*** (a two-dimensional surface) and recording on that surface the points at which a given trajectory cuts through that surface. (In most cases the choice of plane is not crucial as long as the trajectories cut the surface ***transversely***, that is, the trajectories do not run parallel or almost parallel to the surface as they pass through; see Fig. 4.4.) For autonomous systems, such as the Lorenz model equations, we choose some convenient plane in the state space, say, the XY plane for the Lorenz equations. When a trajectory crosses that plane passing from, for example, negative Z values to positive Z values, we record that crossing point.

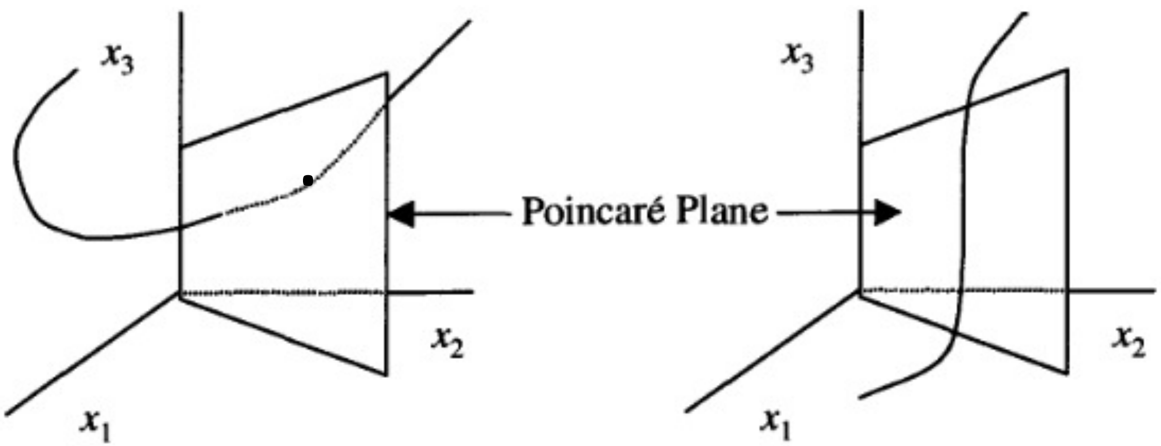


Fig. 4.4. A Poincaré section for a three-dimensional state space. On the left the trajectory crosses the Poincaré plane transversely. On the right the intersection is not transverse because the trajectory runs parallel to the plane for some distance.

In later discussions, it will be useful to indicate on the Poincaré section the record of trajectory intersections with the plane as trajectories approach or diverge from the periodic points. For example, Fig. 4.6 shows a sequence of points P_0, P_1, P_2, . . . as a trajectory approaches an attracting limit cycle in a three-dimensional state space. (Compare Fig. 4.6 with Fig. 3.13.) The reader should be warned that in some diagrams found in the literature this series of dots will be connected with a smooth curve intersecting (x_1^*, x_2^*). It is important to remember that this curve is not a trajectory. In fact the Poincaré intersection of any single trajectory is just a sequence of points as shown in Fig. 4.6. If a smooth curve is drawn on this kind of diagram, it represents the intersection points of an infinite family of trajectories, all of which are approaching (x_1^*, x_2^*). Later we shall see cases in which such curves intersect. It is important to remember that this intersection does not violate the No-Intersection Theorem because the intersecting curves in this case are not themselves trajectories.

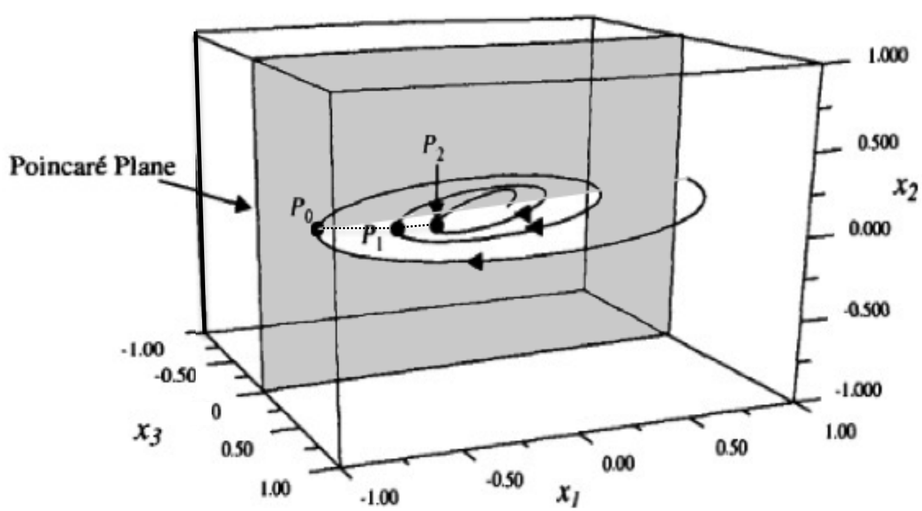
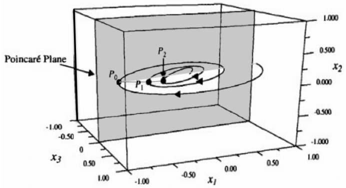


Fig. 4.6. The sequence of points P_0, P_1, P_2, . . . is the record of successive intersections of a single trajectory with the Poincaré plane (the plane with $x_3 = 0$) as the trajectory goes from $x_3 > 0$ to $x_3 < 0$.

MAPPA DI POINCARE' 2D PER LO STUDIO DELLA STABILITA' DEI CICLI LIMITE IN 3D

We now return to the general discussion of limit cycles. The stability of the limit cycle is determined by a generalization of the Poincaré multipliers introduced in the previous chapter. We assume that the uniqueness of the solutions to the equations used to describe the dynamical system entails the existence of a Poincaré map function (or in the present case, a pair of Poincaré map functions), which relate the coordinates of one point at which the trajectory crosses the Poincaré plane to the coordinates of the next (in time) crossing point. (Again we assume we have chosen a definite crossing sense; e.g., from top to bottom, or from left to right.) These functions take the form



$$
\begin{aligned}
x_1^{(n+1)} &= F_1(x_1^{(n)}, x_2^{(n)}) \\
x_2^{(n+1)} &= F_2(x_1^{(n)}, x_2^{(n)})
\end{aligned}
\qquad (4.6\text{-}1)
$$

mappa di Poincarè 2 dim

where the parenthetical superscript indicates the crossing point number.

Here these Poincaré map functions have arisen from the consideration of a Poincaré section for trajectories arising from a set of differential equations. In Chapter 5, we shall consider such map functions as interesting models in their own right, independent of this particular heritage.

The fixed points of the Poincaré section are those points that satisfy

$$
\begin{aligned}
x_1^* &= F_1(x_1^*, x_2^*) \\
x_2^* &= F_2(x_1^*, x_2^*)
\end{aligned}
\qquad (4.6\text{-}2)
$$

Each fixed point in the Poincaré section corresponds to a limit cycle in the full three-dimensional state space.

MAPPA DI POINCARE' 2D PER LO STUDIO DELLA STABILITA' DEI CICLI LIMITE IN 3D

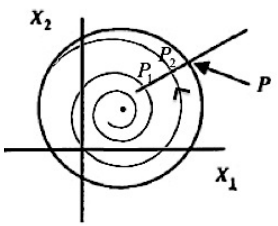
We can characterize the stability of these fixed points by finding the characteristic values of the associated Jacobian matrix of derivatives [sometimes called the ***Floquet matrix***, after Gaston Floquet (1847–1920), a French mathematician who studied, among other things, the properties of differential equations with periodic terms]. This matrix is analogous to the Jacobian matrix used to determine the characteristic values of a fixed point in the full state space. The Jacobian matrix JM is given by

mappa 1 dim

$$d_2 = Md_1$$

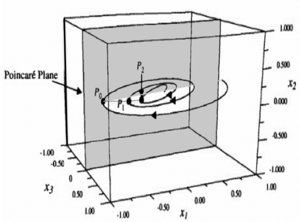
$$d_{n+1} = M^n d_1$$

$$M = \left.\frac{dF}{dP}\right|_{P^*}$$



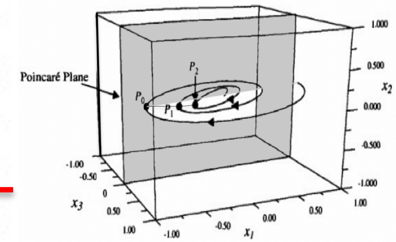
mappa 2 dim

$$JM = \begin{pmatrix} \dfrac{\partial F_1}{\partial x_1} & \dfrac{\partial F_1}{\partial x_2} \\ \dfrac{\partial F_2}{\partial x_1} & \dfrac{\partial F_2}{\partial x_2} \end{pmatrix} \longrightarrow \begin{matrix} M_1 \\ M_2 \end{matrix} \qquad (4.6\text{-}3)$$



where the matrix is to be evaluated at the Poincaré map fixed point in question. The characteristic values of this matrix determine the stability of the limit cycle. A stable limit cycle attracts nearby trajectories, while an unstable limit cycle repels nearby trajectories. In principle, we can use the mathematical methods given in Chapter 3 to find these characteristic values. In practice, however, we most often cannot find these characteristic values explicitly, since, to do that, we would need to know the exact form of the Poincaré map function, and in most cases, we do not know that function. [In Chapter 5, we will examine some models that do give us the map function directly. However, for systems described by differential equations in state spaces of three (or more) dimensions, it is in general impossible to find the map functions.]

Since the Jacobian matrix is a 2×2 matrix for a Poincaré section in a three-dimensional state space, the fixed point has two characteristic values. Hence, we have the same set of stability cases here that we had for fixed points in a two-dimensional state space, with one addition: The intersection points may alternate from one side of the fixed point to the other. (Recall that this alternation was not possible in two dimensions because the trajectory would have to cross itself. In three dimensions the trajectory can wind over and under itself to give the alternation without intersection.)



Dissipation

For a 2×2 matrix, there are two characteristic values. We denote the characteristic values as M_1 and M_2 since we use them as ***Floquet multipliers*** in determining how trajectories approach or diverge from the Poincaré intersection point of the limit cycle. Just as for Poincaré sections in a two-dimensional state space, the criterion for dissipation can be formulated in terms of the multipliers since dissipation is linked to the contraction of clusters of initial conditions. Because M_1, the first multiplier, determines the expansion in the x_1 direction and M_2 the expansion in the x_2 direction, we see that the product $M_1 M_2$ determines the expansion or contraction of *areas* in the Poincaré plane. For a dissipative system, we must have $M_1 M_2 < 1$ on the average (not only near the fixed points). In Chapter 8, we shall consider model map systems that preserve state-space area. They have $M_1 M_2 = 1$.

Stability of Limit Cycles

As we saw in two-dimensional systems, if the fixed point is to be stable and have trajectories in its neighborhood attracted to it, then the *absolute value* of each multiplier must be less than 1. [In state spaces with three or more dimensions, we can have $\boxed{M < 0}$; so the stability criterion is formulated using the absolute value of the multipliers.]

The types of limit cycles are

I. ***Stable limit cycle*** (node for the Poincaré map)
II. ***Repelling limit cycle*** (repellor for the Poincaré map)
III. ***Saddle cycle*** (saddle point for the Poincaré map)

mappa 1 dim

$$d_2 = Md_1$$

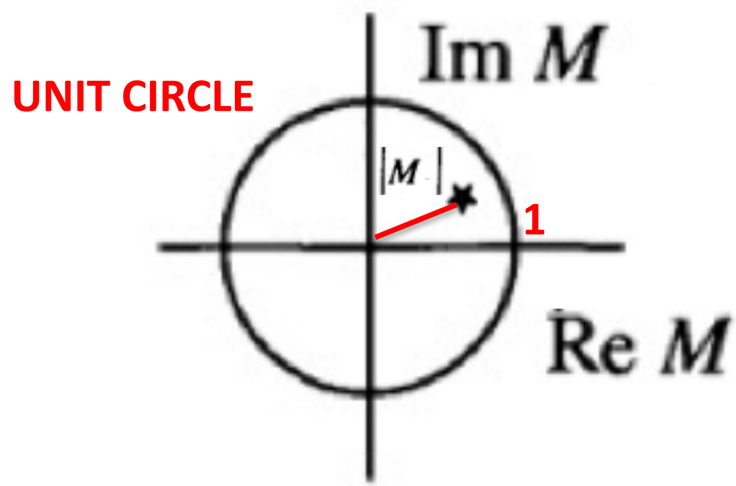
$$d_{n+1} = M^n d_1$$

Table 4.2 lists the categories of characteristic multipliers, the associated Poincaré plane fixed points and the corresponding limit cycles for three-dimensional state spaces. (Compare this table to Table 3.4 for limit cycles in two-dimensional state spaces.)

Table 4.2
Characteristic Multipliers for Poincaré Sections of Three-Dimensional State Spaces

Type of Fixed Point	Characteristic Multiplier	Corresponding Cycle
Node	$\lvert M_1 \rvert, \lvert M_2 \rvert < 1$	Limit Cycle
Repellor	$\lvert M_1 \rvert, \lvert M_2 \rvert > 1$	Repelling Cycle
Saddle	$\lvert M_1 \rvert < 1, \lvert M_2 \rvert > 1$	Saddle Cycle

Of course, the characteristic multipliers could also be complex numbers. Just as we saw for fixed points in a two-dimensional state space, the complex multipliers will form a complex-conjugate pair. In more graphic terms, the successive Poincaré intersection points associated with complex-valued multipliers rotate around the limit cycle intersection point as they approach or diverge from that point. Mathematically, the condition for stability is still the same: the absolute value of both multipliers must be less than 1 for a stable limit cycle. In terms of the corresponding Argand diagram (complex mathematical plane), both characteristic values must lie within a circle of unit radius (called the ***unit circle***) for a stable limit cycle. See Fig. 4.7. As a control parameter is changed the values of the characteristic multipliers can change. If at least one of the characteristic multipliers crosses the unit circle, a bifurcation occurs. Some of these bifurcations will be discussed in the latter part of this chapter.



Stable Limit Cycles

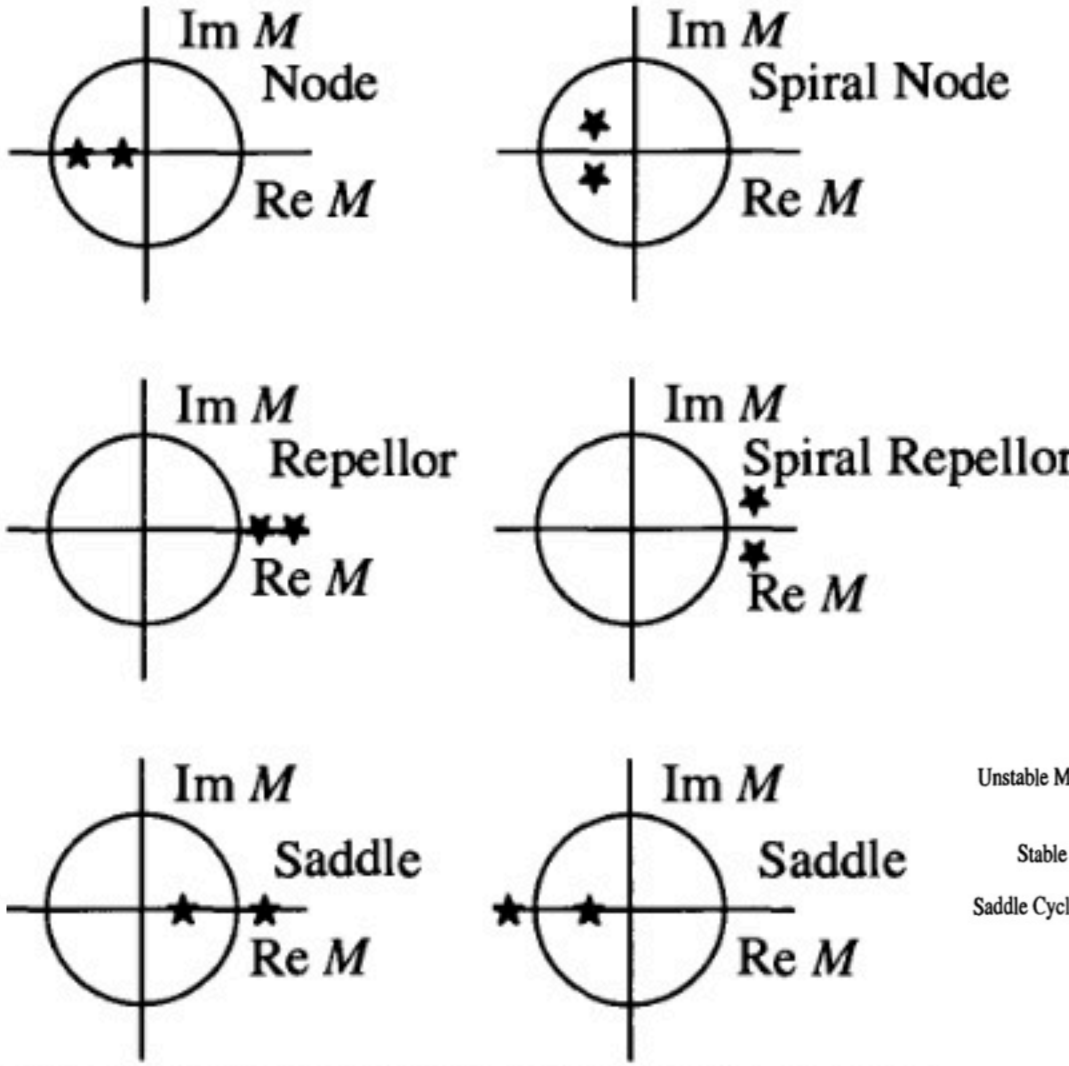
$|M_1|, |M_2| < 1$

Repelling Limit Cycles

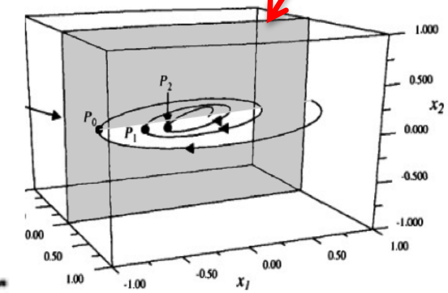
$|M_1|, |M_2| > 1$

Saddle Limit Cycles

$|M_1| < 1, |M_2| > 1$



Sezione di Poincarè



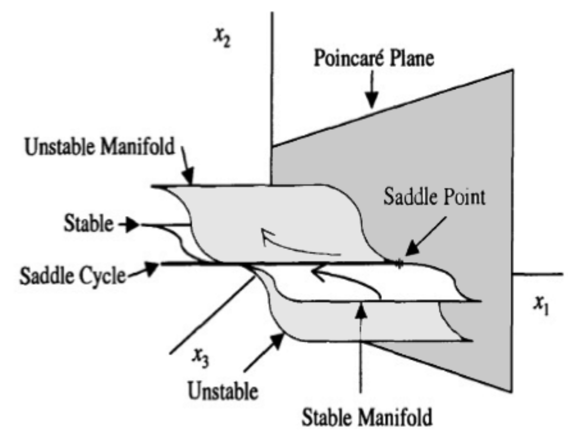


Fig. 4.7. Characteristic multipliers in the complex plane. If both multipliers lie within a circle of unit radius (the unit circle), then the corresponding limit cycle is stable. If one (or both) of the multipliers lies outside the unit circle, then the limit cycle is unstable.

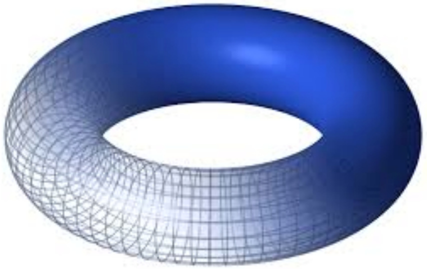
4.7 Quasi-Periodic Behavior (dim = 2)

For a three-dimensional state space, a new type of motion can occur, a type of motion not possible in one- or two-dimensional state spaces. This new type of motion is called *quasi-periodic* because it has two different frequencies associated with it; that is, it can be analyzed into two independent, periodic motions. For quasi-periodic motion, the trajectories are constrained to the surface of a torus in the three-dimensional state space. A mathematical description of this kind of motion is given by:

NON SONO EQUAZIONI DIFFERENZIALI!

EQ. TRAIETTORIA $x_1(t)$, $x_2(t)$ e $x_3(t)$

$$\left\{\begin{aligned} x_1 &= (R + r\sin\omega_r t)\cos\omega_R t \\ x_2 &= r\cos\omega_r t \\ x_3 &= (R + r\sin\omega_r t)\sin\omega_R t \end{aligned}\right. \qquad (4.7\text{-}1)$$



where the two angular frequencies are denoted by ω_R and ω_r. Geometrically, Eqs. (4.7-1) describe motion on the surface of a torus (with the center of the torus at the origin), whose large radius is R and whose cross-sectional radius is r. In general the torus (or doughnut-shape or the shape of the inner tube of a bicycle tire) will look something like Fig. 4.8. The frequency ω_R corresponds to the rate of rotation around the large circumference with a period $T_R = 2\pi/\omega_R$, while the frequency ω_r corresponds to the rate of rotation about the cross section with $T_r = 2\pi/\omega_r$. A general torus might have elliptical cross sections, but the ellipses can be made into circles by suitably rescaling the coordinate axes.

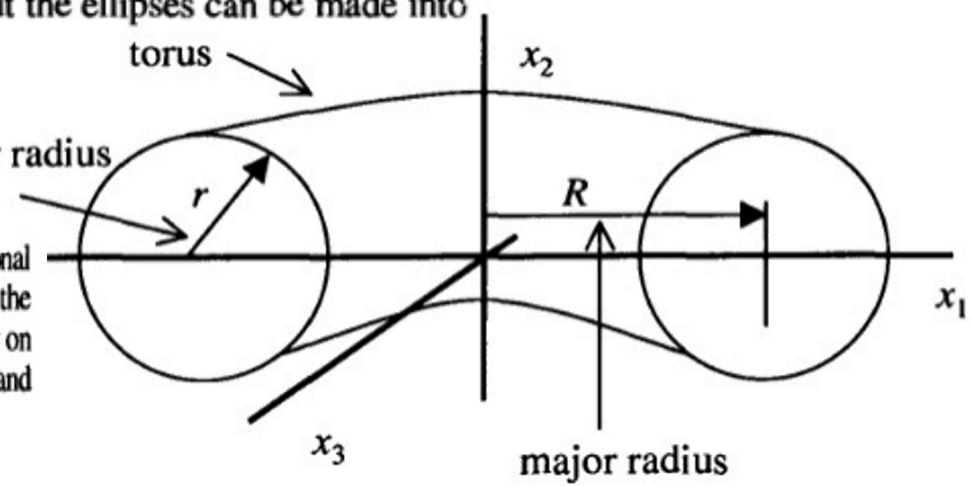
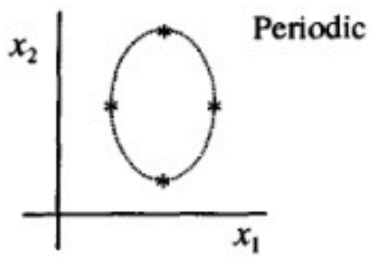


Fig. 4.8. Quasi-periodic trajectories roam over the surface of a torus in three-dimensional state space. Illustrated here is the special case of a torus with circular cross sections. r is the minor radius of the cross section. R is the major radius of the torus. A periodic trajectory on the surface of the torus would close on itself. On the right. a perspective view of a torus and a Poincaré plane.

The Poincaré section for this motion is generated by using a Poincaré plane that cuts through the torus. What the pattern of Poincaré map points looks like depends on the numerical relationship between the two frequencies as illustrated in Fig. 4.9. If the ratio of the two frequencies can be expressed as the ratio of two integers (that is, as a "rational fraction," 14/17, for example), then the Poincaré section will consist of a finite number of points. This type of motion is often called ***frequency-locked*** motion because one of the frequencies is locked, often over a finite control parameter range, so that an integer multiple of one frequency is equal to another integer multiple of the other. (The terms ***phase-locking*** and ***mode-locking*** are also used to describe this behavior.)

$\dfrac{\omega_R}{\omega_r}$ razionale



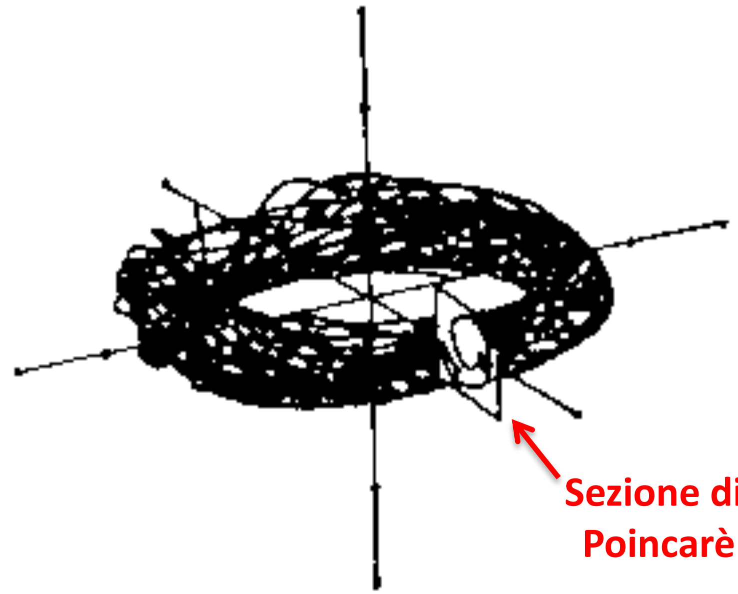
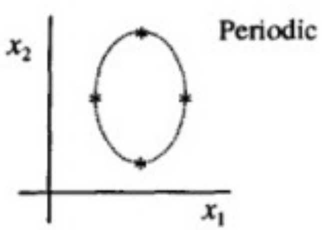


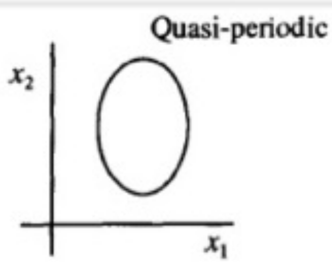
Fig. 4.9. A Poincaré section intersects a torus in three–dimensional state space. The diagram on the upper left shows the Poincaré map points for a two-frequency periodic system with a rational ratio of frequencies. The intersection points are indicated by asterisks. The diagram on the lower left is for quasi-periodic behavior. The ratio of frequencies is irrational, and eventually the intersection points fill in a curve (sometimes called a "drift ring") in the Poincaré plane.

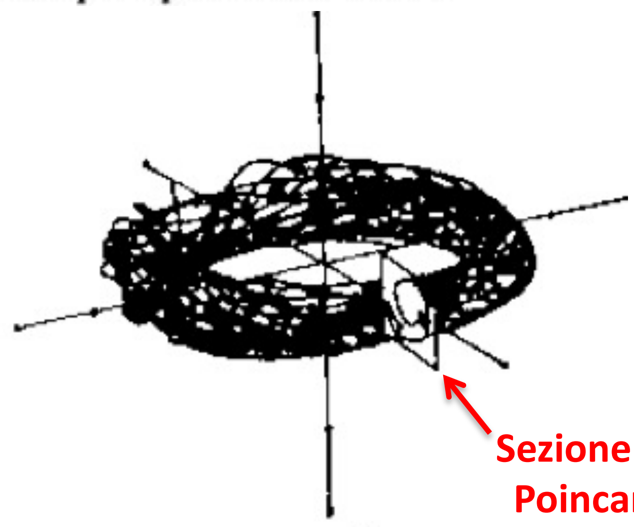
If the ratio of frequencies cannot be expressed as a ratio of integers, then the ratio is called "irrational" (in the mathematical, not the psychological sense). For the irrational case, the Poincaré map points will eventually fill in a continuous curve in the Poincaré plane, and the motion is said to be ***quasi-periodic*** because the motion never exactly repeats itself. (Russian mathematicians call this ***conditionally periodic***. See, for example, [Arnold, 1983]. The term ***almost periodic*** is also used in the mathematical literature.)

In the quasi-periodic case the motion, strictly speaking, never exactly repeats itself (hence, the modifier *quasi*), but the motion is not chaotic; it is composed of two (or more) periodic components, whose presence could be made known by measuring the frequency spectrum (Fourier power spectrum) of the motion. We should point out that detecting the difference between quasi-periodic motion and motion with a rational ratio of frequencies, when the integers are large, is a delicate question. Whether a given experiment can distinguish the two cases depends on the resolution of the experimental equipment. As we shall see later, the behavior of the system can switch abruptly back and forth between the two cases as a parameter of the system is varied. The important point is that the attractor for the system is a two-dimensional surface of the torus for quasi-periodic behavior.

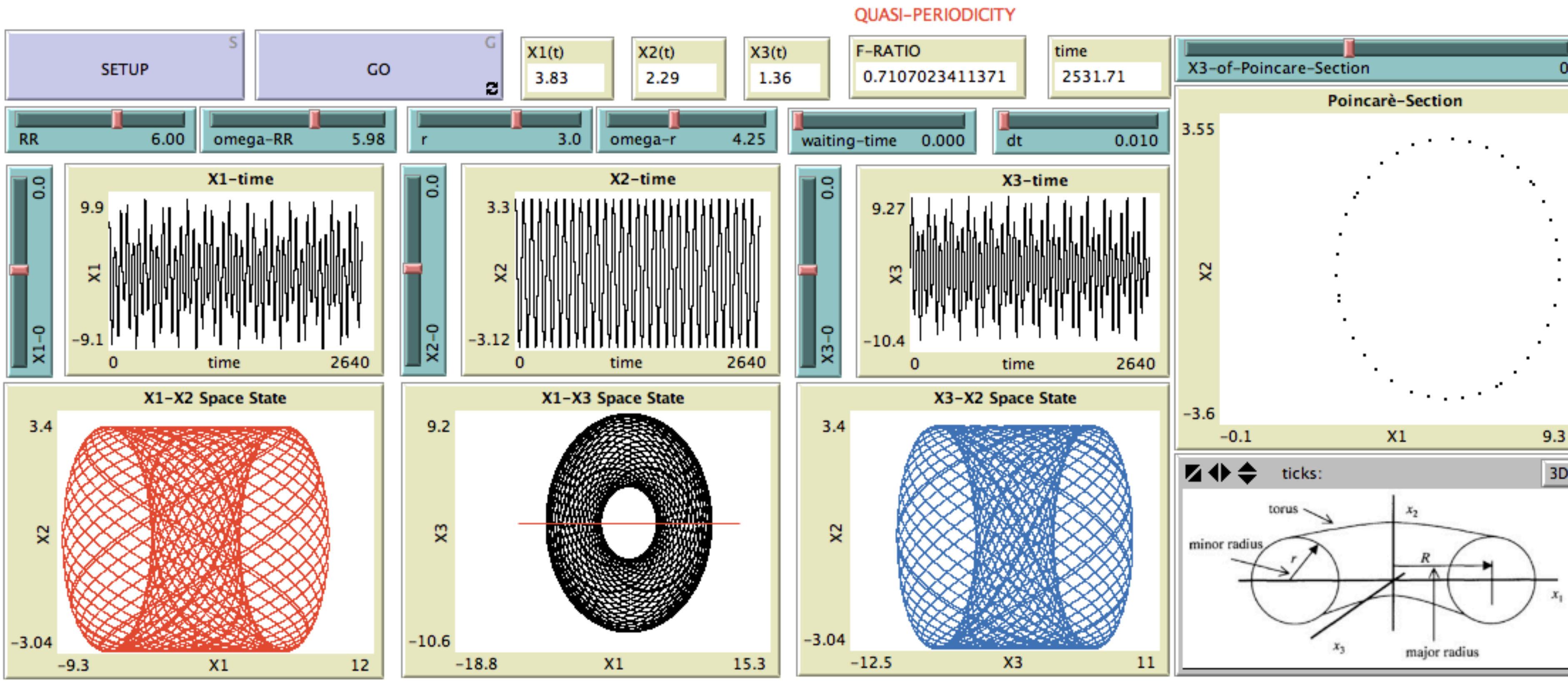


$\frac{\omega_R}{\omega_r}$ **irrazionale**





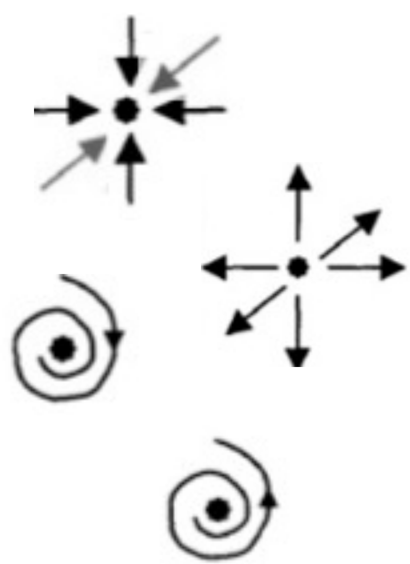
quasi-periodicity.nlogo



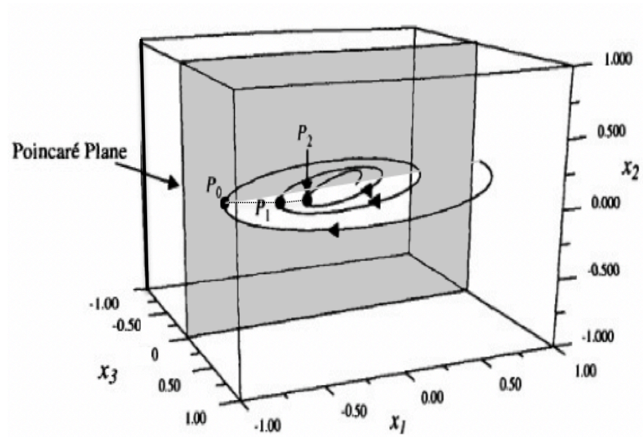
We have now seen the full panoply of regular (nonchaotic) attractors: fixed points (dimension 0), limit cycles (dimension 1), and quasi-periodic attractors (dimension 2 or more). We are ready to begin the discussion of how these attractors can change into chaotic attractors.

We will give only a brief discussion of the period-doubling, quasi-periodic, and intermittency routes. These will be discussed in detail in Chapters 5, 6, and 7, respectively. A discussion of crises will be found in Chapter 7. As we shall see, the chaotic transient route is more complicated to describe because it requires a knowledge of what trajectories are doing over a range of state space. We can no longer focus our attention locally on just a single fixed point or limit cycle.

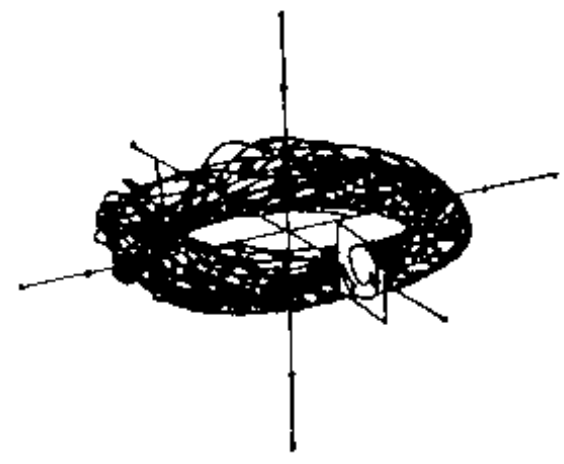
fixed points (dim.0)

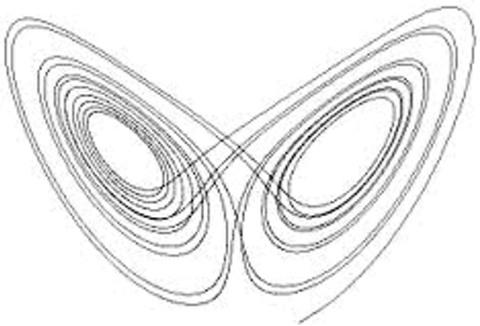


limit cycles (dim.1)



quasiperiodic attractors (dim.2)





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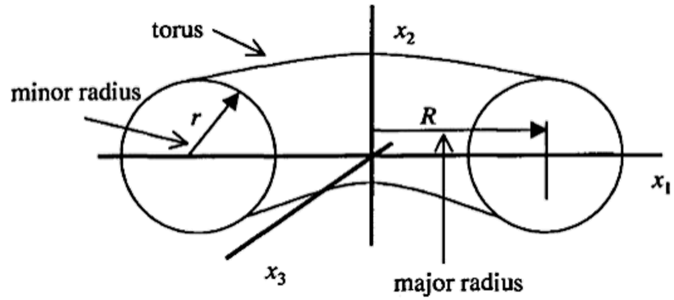
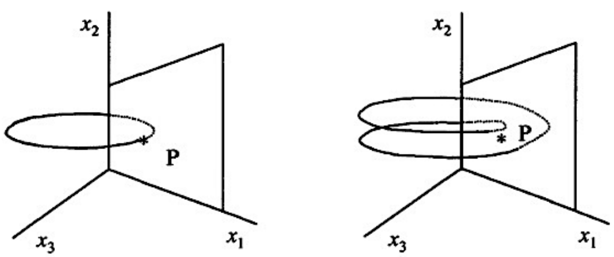
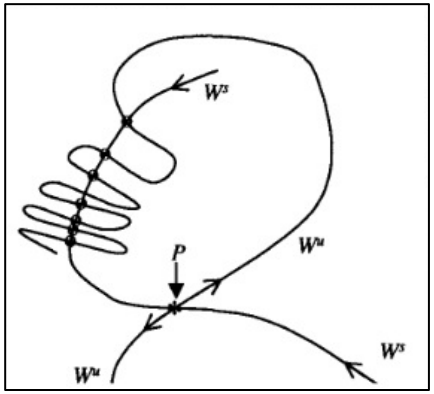
II : Quasi-Periodicity



I : Period-Doubling



IV : Chaotic Transient and Homoclinic Orbits



III : Intermittency and Crises