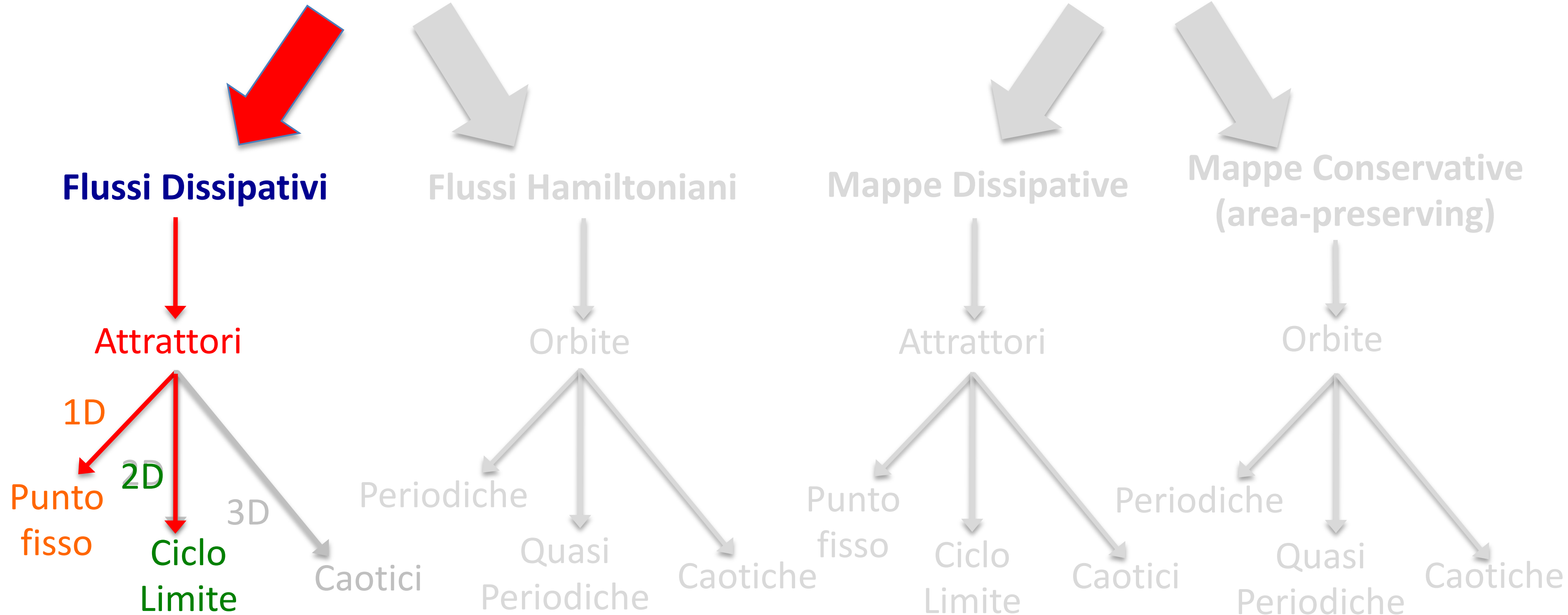
Classificazione dei Sistemi Dinamici

Sistemi dinamici continui (Flussi)

$$\dot{X} = f(X)$$

Sistemi dinamici discreti (Mappe)

$$x_{n+1} = f(x_n)$$

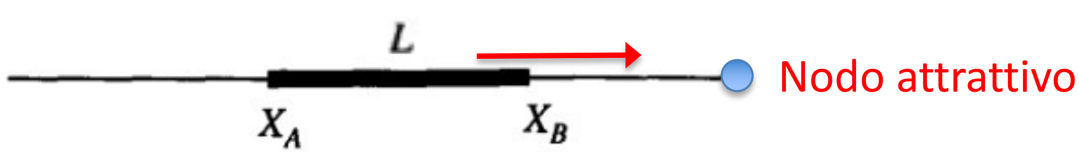


$$\dot{X} = f(X)$$

Flussi dissipativi in una dimensione

$$\frac{1}{L}\frac{dL}{dt} = \frac{1}{L}\left[f(X_B) - f(X_A)\right] = \frac{df(X)}{dX} < 0$$

fixed points (dim.0)

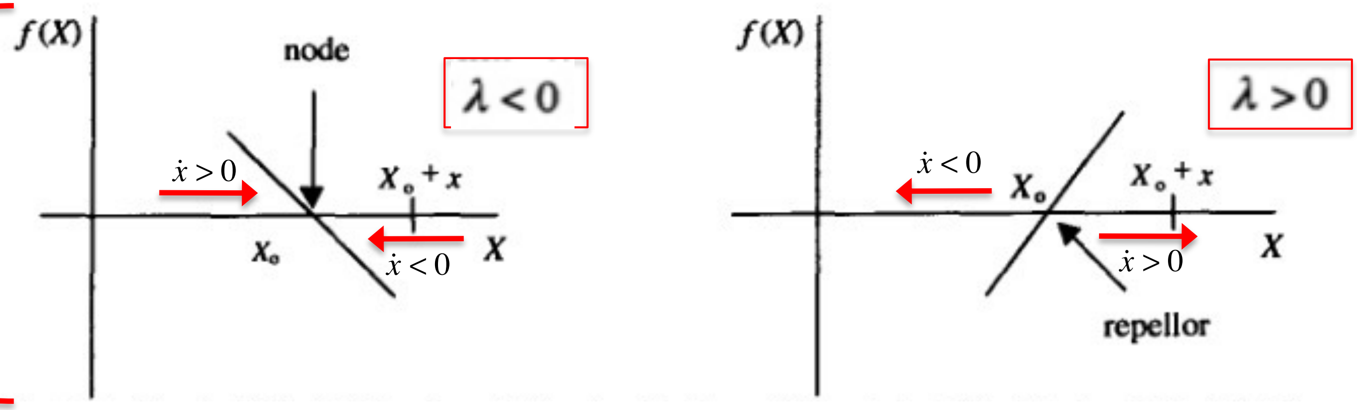


A "cluster of initial conditions," indicated by the heavy line, along the X axis.

Riepilogo dei Punti Fissi in uno Spazio degli Stati a Una Dimensione

$$\dot{X}\Big|_{X=X_o} = f(X_o) = 0$$

Punti Fissi Strutturalmente Stabili



$$\lambda = \frac{df(X)}{dX}\Big|_{X=X_o}$$

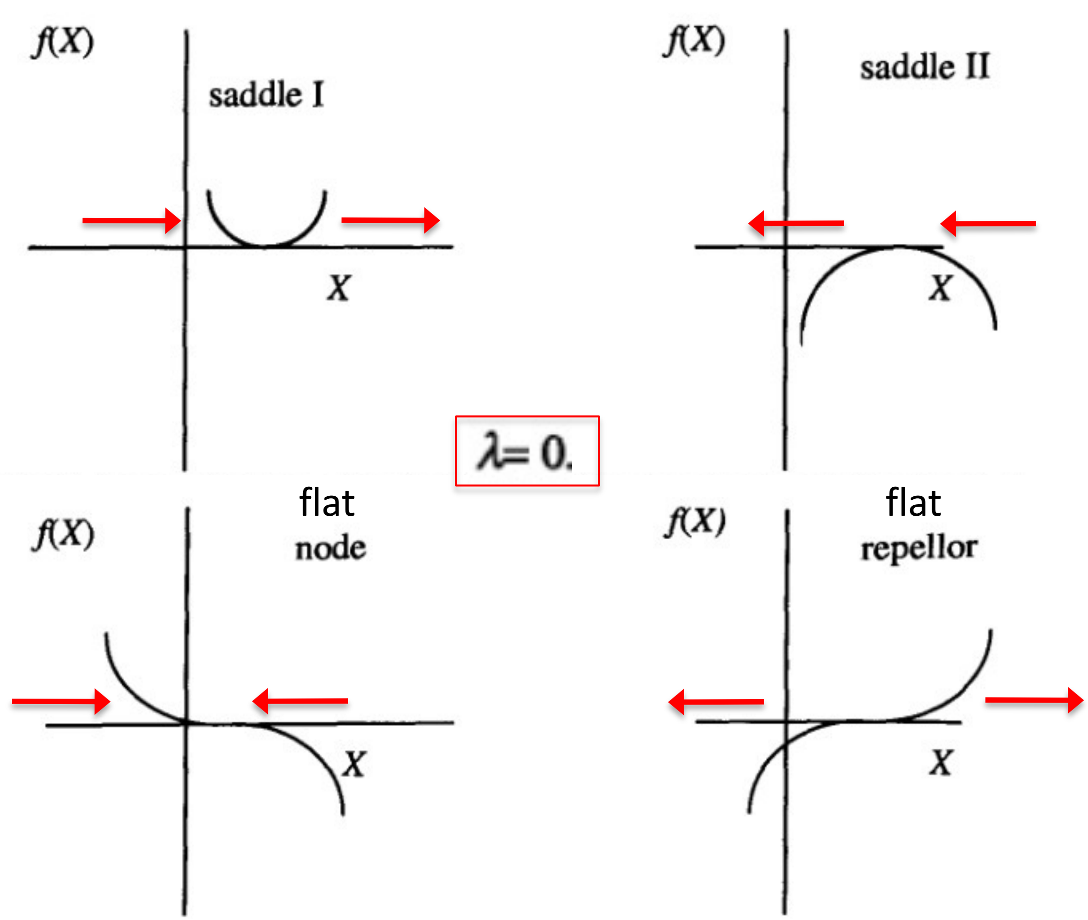
Valore Caratteristico del Punto Fisso

Punti Fissi Strutturalmente Instabili

$$\dot{x} = \frac{df}{dX}\Big|_{X_o} x$$

Equazione linearizzata della distanza dal Punto Fisso

$$x(t) = x(0)e^{\lambda t}$$



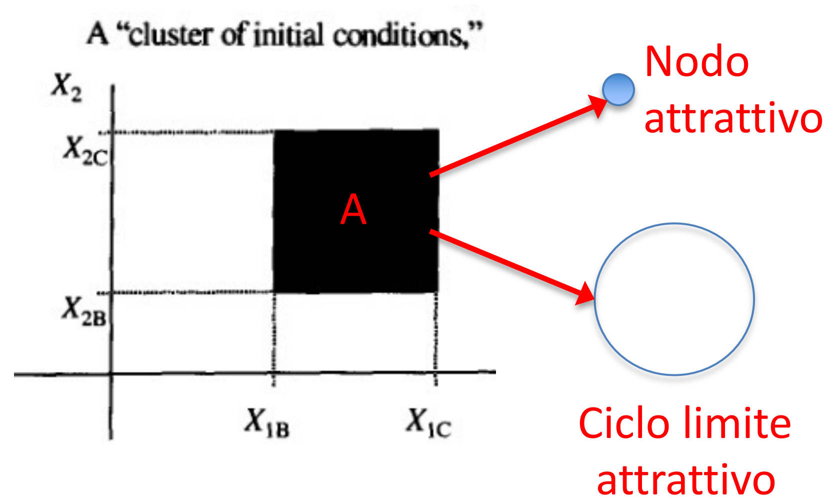
$$\dot{X}_1 = f_1(X_1, X_2)$$
$$\dot{X}_2 = f_2(X_1, X_2)$$

Flussi dissipativi in due dimensioni

$$\frac{1}{A}\frac{dA}{dt} = \frac{\partial f_1}{\partial X_1} + \frac{\partial f_2}{\partial X_2} < 0$$

fixed points (dim.0)

limit cycles (dim.1)



Studio della stabilità dei Punti Fissi in due dimensioni: il caso generale

$$\dot{X}_1 = f_1(X_1, X_2)$$
$$\dot{X}_2 = f_2(X_1, X_2)$$

$$\begin{cases} f_1(X_{1o}, X_{2o}) = 0 \\ f_2(X_{1o}, X_{2o}) = 0 \end{cases}$$

→ FIXED POINTS

PER CIASCUN PUNTO FISSO (X_{1o}, X_{2o}) :

$$\dot{X}_1 = f_1(X_1, X_2) = (X_1 - X_{1o})\frac{\partial f_1}{\partial X_1} + (X_2 - X_{2o})\frac{\partial f_1}{\partial X_2} + \ldots \qquad (3.11\text{-}4a)$$

DISTANZA DELLA TRAIETTORIA DAL PUNTO FISSO LUNGO L'ASSE X_1

DISTANZA DELLA TRAIETTORIA DAL PUNTO FISSO LUNGO L'ASSE X_2

$$\dot{X}_2 = f_2(X_1, X_2) = (X_1 - X_{1o})\frac{\partial f_2}{\partial X_1} + (X_2 - X_{2o})\frac{\partial f_2}{\partial X_2} + \ldots \qquad (3.11\text{-}4b)$$

and ignoring all the higher-order derivative terms, we may write Eq. (3.11-4) as

Equazioni linearizzate attorno al punto fisso

$$\begin{cases} \dot{x}_1 = \dfrac{\partial f_1}{\partial x_1} x_1 + \dfrac{\partial f_1}{\partial x_2} x_2 \\ \dot{x}_2 = \dfrac{\partial f_2}{\partial x_1} x_1 + \dfrac{\partial f_2}{\partial x_2} x_2 \end{cases} \qquad (3.11\text{-}6)$$

Studio della stabilità dei Punti Fissi in due dimensioni: il caso generale

$$\dot{X}_1 = f_1(X_1, X_2)$$
$$\dot{X}_2 = f_2(X_1, X_2)$$

$$\begin{cases} f_1(X_{1o}, X_{2o}) = 0 \\ f_2(X_{1o}, X_{2o}) = 0 \end{cases} \longrightarrow \text{FIXED POINTS}$$

PER CIASCUN PUNTO FISSO (X_{1o}, X_{2o}) **:**

Equazione Caratteristica

$$\lambda^2 - (f_{11} + f_{22})\lambda + (f_{11}f_{22} - f_{12}f_{21}) = 0 \qquad (3.11\text{-}11)$$

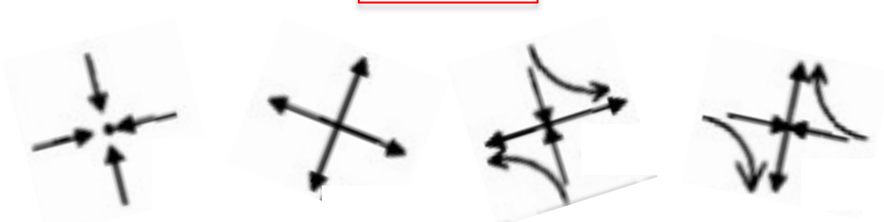
We call Eq. (3.11-11) the ***characteristic equation*** for λ, whose value depends only on the derivatives of the time evolution functions evaluated at the fixed point. Eq. (3.11-11) is a quadratic equation for λ and in general has two solutions, which we can write down from the standard quadratic formula:

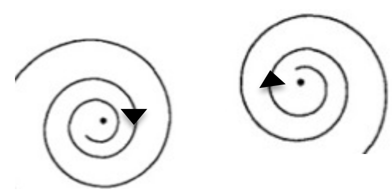
2 Valori caratteristici

$$\lambda_{\pm} = \frac{f_{11} + f_{22} \pm \sqrt{(f_{11} + f_{22})^2 - 4(f_{11}f_{22} - f_{12}f_{21})}}{2} \qquad (3.11\text{-}12)$$

reali $\lambda_{\pm} = R$

complessi coniugati $\lambda_{\pm} = R \pm i\Omega$





Studio della stabilità dei Punti Fissi in due dimensioni: il caso generale

For the fixed point on the left in Fig. 3.10, we say we have a ***spiral node*** (sometimes called a ***focus***) since the trajectories spiral in toward the fixed point. On the right in Fig. 3.10, we have a ***spiral repellor*** (sometimes called an ***unstable focus***). In the special case when $R = 0$, the trajectory forms a closed loop around the fixed point. This closed loop trajectory is called a ***cycle***. If trajectories in the neighborhood of this cycle are attracted toward it as time goes on, then the cycle is called a ***limit cycle***. We need a more detailed analysis to see if this cycle is itself stable or unstable. An analysis of cycle behavior will be taken up in Section 3.16.

It is important to realize that the spiral type behavior shown in Fig. 3.10 and the cycle type behavior discussed in the Section 3.16 are possible only in state spaces of two (or higher) dimensions. They cannot occur in a one-dimensional state space because of the No-Intersection Theorem (recall Exercise 3.8-2).

Distanze del punto rappresentativo del sistema dal punto fisso lungo i due assi dello spazio degli stati:

$$\lambda_\pm = R \pm i\Omega$$

$$\begin{cases} x_1(t) = F_1 e^{Rt} \sin \Omega t \\ x_2(t) = F_2 e^{Rt} \sin \Omega t \end{cases}$$

con:

$$F_1 \approx \frac{f_{12} x_2(0)}{\Omega}$$

$$F_2 \approx \frac{f_{21} x_1(0)}{\Omega}$$

$$R = \frac{1}{2}(f_{11} + f_{22})$$

$$\Omega = \frac{1}{2}\sqrt{\left|(f_{11} + f_{22})^2 - 4(f_{11} f_{22} - f_{12} f_{21})\right|}$$

Studio della stabilità dei Punti Fissi in due dimensioni: il caso generale

For the fixed point on the left in Fig. 3.10, we say we have a ***spiral node*** (sometimes called a ***focus***) since the trajectories spiral in toward the fixed point. On the right in Fig. 3.10, we have a ***spiral repellor*** (sometimes called an ***unstable focus***). In the special case when $R = 0$, the trajectory forms a closed loop around the fixed point. This closed loop trajectory is called a ***cycle***. If trajectories in the neighborhood of this cycle are attracted toward it as time goes on, then the cycle is called a ***limit cycle***. We need a more detailed analysis to see if this cycle is itself stable or unstable. An analysis of cycle behavior will be taken up in Section 3.16.

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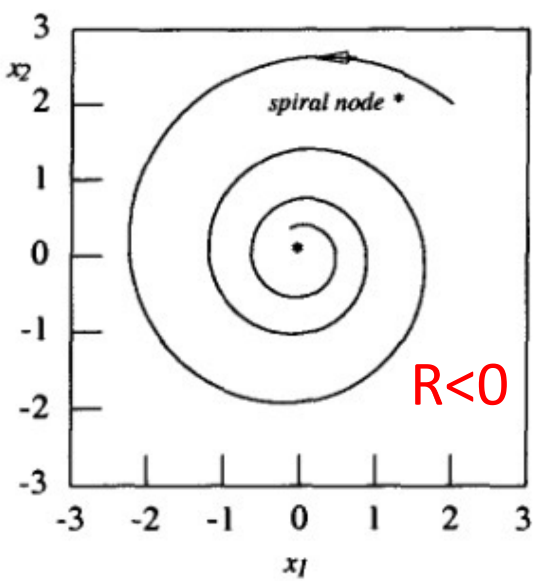
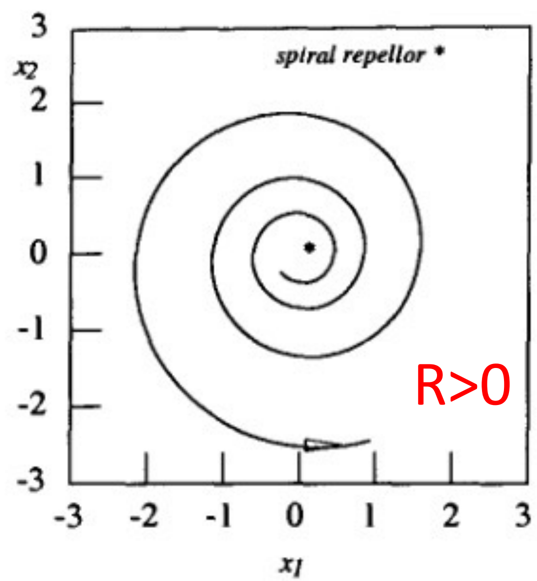
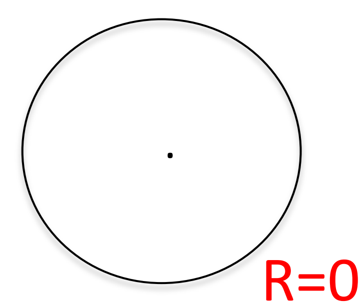




CICLO LIMITE



Metodo dello Jacobiano per studiare i punti fissi nel caso generale a 2 dim.

Equazioni originarie

$$\dot{X}_1 = f_1(X_1, X_2)$$
$$\dot{X}_2 = f_2(X_1, X_2)$$

...ricavare i punti fissi...

Equazioni linearizzate nelle vicinanze di un dato punto fisso (X_{1o}, X_{2o})

$$\dot{x}_1 = \frac{\partial f_1}{\partial x_1} x_1 + \frac{\partial f_1}{\partial x_2} x_2$$
$$\dot{x}_2 = \frac{\partial f_2}{\partial x_1} x_1 + \frac{\partial f_2}{\partial x_2} x_2$$

with $f_{ij} = \frac{\partial f_i}{\partial x_j}$

...calcolate nel punto fisso

Distanze dal punto fisso

$$x_1 = X_1 - X_{1o}$$
$$x_2 = X_2 - X_{2o}$$

3.14 The Jacobian Matrix for Characteristic Values

We would now like to introduce a more elegant and general method of finding the characteristic equation for a fixed point. This method makes use of the so-called ***Jacobian matrix*** of the derivatives of the time evolution functions. Once we see how this procedure works, it will be easy to generalize the method, at least in principle, to find characteristic values for fixed points in state spaces of any dimension. The Jacobian matrix for the system is defined to be the following square array of the derivatives:

Matrice Jacobiana

$$J = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \quad \rightarrow \quad \lambda_+, \lambda_- \qquad (3.14\text{-}1)$$

Autovalori

where the derivatives are evaluated at the fixed point. We subtract λ from each of the principal diagonal (upper left to lower right) elements and set the determinant of the matrix equal to 0:

Metodo dello Jacobiano per studiare i punti fissi nel caso generale a 2 dim.

Eq. agli autovalori

$$J = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \qquad \boxed{J\vec{v} = \lambda\vec{v}} \rightarrow \det(J - \lambda I) = 0 \rightarrow \begin{vmatrix} f_{11} - \lambda & f_{12} \\ f_{21} & f_{22} - \lambda \end{vmatrix} = 0$$

Equazione caratteristica dello Jacobiano

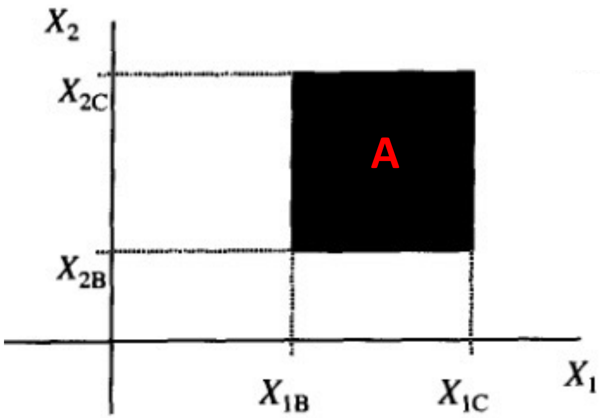
$$\lambda^2 - (f_{11} + f_{22})\lambda + (f_{11}f_{22} - f_{12}f_{21}) = 0 \qquad (3.11\text{-}11)$$

Autovalori dello Jacobiano

$$\lambda_{\pm} = \frac{f_{11} + f_{22} \pm \sqrt{(f_{11} + f_{22})^2 - 4(f_{11}f_{22} - f_{12}f_{21})}}{2} \qquad (3.11\text{-}12)$$

Multiplying out the determinant in the usual way then yields the characteristic equation (3.11-11). The Jacobian matrix method is obviously easily extended to *d*-dimensions by writing down the *d*-by-*d* matrix of derivatives of the *d* time-evolution functions f_n, forming the corresponding determinant, and then (at least in principle) solving the resulting *d*th order equation for the characteristic values.

We now introduce some terminology from linear algebra to make some very general and very powerful statements about the characteristic values for a given fixed point.



Reminder: condizione affinchè un cluster di condizioni iniziali collassi su un attrattore stabile

$$\frac{1}{A}\frac{dA}{dt} = (f_{11} + f_{22}) < 0 \longrightarrow \boxed{TrJ < 0}$$

First, the ***trace*** of a matrix, such as the Jacobian matrix (3.14-1), is defined to be the sum of the principal diagonal elements. For Eq. (3.14-1) this is explicitly

Traccia dello Jacobiano

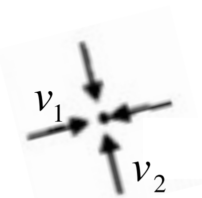
$$TrJ = f_{11} + f_{22} \tag{3.14-3}$$

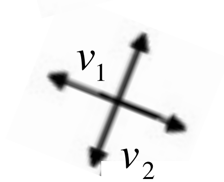
$$x_1(t) = F_1 e^{Rt} \sin \Omega t$$
$$x_2(t) = F_2 e^{Rt} \sin \Omega t$$
$$R = \frac{1}{2}(f_{11} + f_{22})$$

According to Eq. (3.13-5), however, this is just the combination of derivatives needed to test whether or not the system's trajectories collapse toward an attractor. To make a connection with the previous section, we note that $TrJ = 2R$, so that we see that the sign of TrJ determines whether the fixed point is a node or a repellor.

Linear algebra also tells us how to find the directions to be associated with the characteristic values.

In linear algebra this procedure is called "finding the eigenvalues and eigenvectors of the matrix." For our purposes, the eigenvalues are the characteristic values of the fixed point and the eigenvectors give the associated characteristic directions. However, we will not need these eigenvectors for most of our purposes. The interested reader is referred to the books on linear algebra listed at the end of the chapter.





$$J = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

We now introduce one more symbol:

Determinante dello Jacobiano: $\Delta = f_{11}f_{22} - f_{21}f_{12}$ (3.14-6)

Δ is called the ***determinant*** of that matrix. Then we may show that the nature of the fixed point is determined by TrJ and Δ as listed in Table 3.3.

$$\lambda_{\pm} = \frac{f_{11} + f_{22} \pm \sqrt{(f_{11} + f_{22})^2 - 4(f_{11}f_{22} - f_{12}f_{21})}}{2} \rightarrow \lambda_{\pm} = \frac{TrJ \pm \sqrt{(TrJ)^2 - 4\Delta}}{2}$$

$$\lambda_{\pm} = R \pm i\Omega$$

$$x_1(t) = F_1 e^{Rt} \sin \Omega t$$

$$x_2(t) = F_2 e^{Rt} \sin \Omega t$$

$$\begin{cases} R = \frac{1}{2} TrJ \\ \Omega = \frac{1}{2}\sqrt{TrJ^2 - 4\Delta} \end{cases}$$

Table 3.3

Fixed Points for Two-dimensional State Space

	$TrJ < 0$	$TrJ > 0$
$\Delta > (1/4)(TrJ)^2$	spiral node	spiral repellor
$0 < \Delta < (1/4)(TrJ)^2$	node	repellor
$\Delta < 0$	saddle point	saddle point

Riepilogo dei Punti Fissi in uno Spazio degli Stati a Due Dimensioni

$$\lambda_{\pm} = \frac{TrJ \pm \sqrt{(TrJ)^2 - 4\Delta}}{2}$$

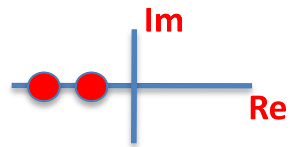
con: $\begin{cases} TrJ = f_{11} + f_{22} \\ \Delta = f_{11}f_{22} - f_{21}f_{12} \end{cases}$

$\Delta > 0$

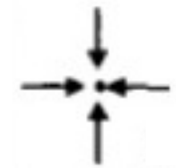
- $(TrJ)^2 - 4\Delta > 0$

  $0 < \Delta < \frac{1}{4}(TrJ)^2 \quad \rightarrow \lambda_+, \lambda_-$ **reali e concordi**

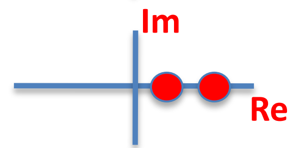
  - $TrJ < 0$

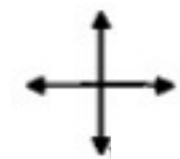
    NODE

  - $TrJ > 0$

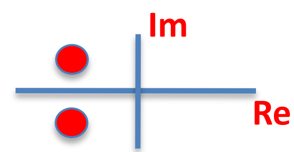
    REPELLOR

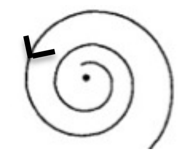
- $(TrJ)^2 - 4\Delta < 0$

  $\Delta > \frac{1}{4}(TrJ)^2 \quad \rightarrow \lambda_+, \lambda_-$ **complessi coniugati**

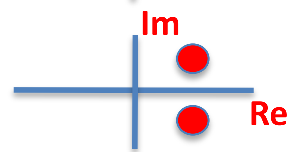
  - $TrJ < 0$

    SPIRAL NODE

  - $TrJ > 0$

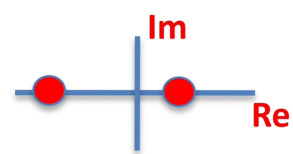
    

    SPIRAL REPELLOR

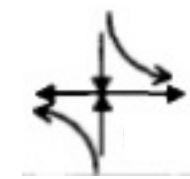
    

$\Delta < 0 \quad \rightarrow \lambda_+, \lambda_-$ **reali e discordi**

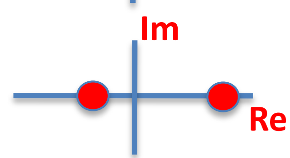
- $TrJ < 0$

  SADDLE POINT

- $TrJ > 0$

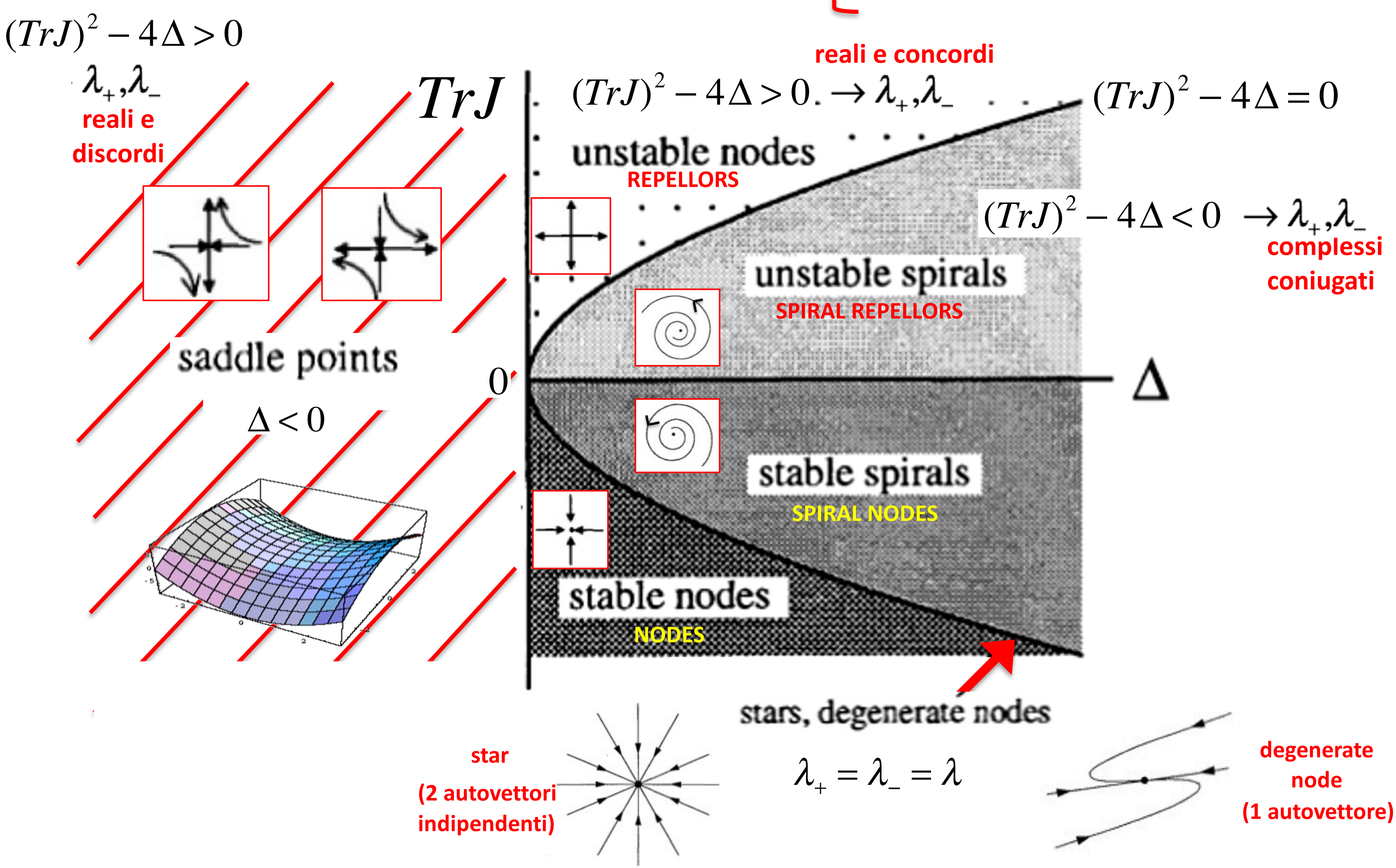
  SADDLE POINT

Diagramma dei Punti Fissi in uno Spazio degli Stati a Due Dimensioni

$$\lambda_{\pm} = \frac{TrJ \pm \sqrt{(TrJ)^2 - 4\Delta}}{2}$$

con:
$$\begin{cases} TrJ = f_{11} + f_{22} \\ \Delta = f_{11}f_{22} - f_{21}f_{12} \end{cases}$$



Summary of Fixed Point Analysis for Two-dimensional State Space

1. Write the time evolution equations in the first-order time derivative form of Eq. (3.10-1).

$$\begin{aligned} \dot{X}_1 &= f_1(X_1, X_2) \\ \dot{X}_2 &= f_2(X_1, X_2) \end{aligned} \tag{3.10-1}$$

2. Find the fixed points of the evolution by finding those points that satisfy

$$\begin{aligned} f_1(X_1, X_2) &= 0 \\ f_2(X_1, X_2) &= 0 \end{aligned}$$

3. At the fixed points, evaluate the partial derivatives of the time evolution functions to set up the Jacobian matrix

$$J \equiv \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \tag{3.14-1}$$

4. Evaluate the trace and determinant of the Jacobian matrix at the fixed point and use Table 3.3 to find the type of fixed point.
5. Use Eq. (3.11-12) to find the numerical values of the characteristic values and to specify the behavior of the state-space trajectories near the fixed point with Eq. (3.11-13).

Examples by Topic

Matrices & Linear Algebra

» PRO: Data Input
» PRO: Image Input
» PRO: File Upload
» PRO: CDF Interactivity
» Mathematics
» Statistics & Data Analysis
» Physics
» Chemistry
» Materials
» Engineering
» Astronomy
» Earth Sciences
» Life Sciences
» Computational Sciences
» Units & Measures
» Dates & Times
» Weather
» Places & Geography
» People & History
» Culture & Media
» Music
» Words & Linguistics
» Sports & Games
» Colors
» Shopping

Matrix Arithmetic

do basic arithmetic on matrices

{{0,-1},{1,0}}.{{1,2},{3,4}}+{{2,-1},{-1,2}}

{{2,-1,1},{0,-2,1},{1,-2,0}}.{x,y,z}

Matrix Operations

compute properties of a matrix

{{6, -7, 10}, {0, 3, -1}, {0, 5, -7}}

compute the rank of a matrix

rank {{6, -11, 13}, {4, -1, 3}, {3, 4, -2}}

compute the inverse of a matrix

inv {{10, -9, -12}, {7, -12, 11}, {-10, 10, 3}}

inverse {{a, b}, {c, d}}

{{2,3},{4,5}}^(-1)

compute the adjugate of a matrix

adjugate {{8,7,7},{6,9,2},{-6,9,-2}}

Trace

compute the trace of a matrix

tr {{9, -6, 7}, {-9, 4, 0}, {-8, -6, 4}}

Determinant

compute the determinant of a matrix

determinant of {{3,4},{2,1}}

det({{9, 3, 5}, {-6, -9, 7}, {-1, -8, 1}})

det {{a, b, c}, {d, e, f}, {g, h, j}}

Row Reduction

row reduce a matrix

row reduce {{2,1,0,-3},{3,-1,0,1},{1,4,-2,-5}}

row reduction calculator

Eigenvalues & Eigenvectors

compute the eigenvalues of a matrix

eigenvalues {{4,1},{2,-1}}

compute the eigenvectors of a matrix

eigenvectors {{1,0,0},{0,0,1},{0,1,0}}

compute the characteristic polynomial of a matrix

characteristic polynomial {{4,1},{2,-1}}

Diagonalization

diagonalize a matrix

diagonalize {{1,2},{3,4}}

Matrix Decompositions »

compute the LU decomposition of a square matrix

LU decomposition of {{7,3,-11},{-6,7,10},{-11,2,-2}}

compute a singular value decomposition

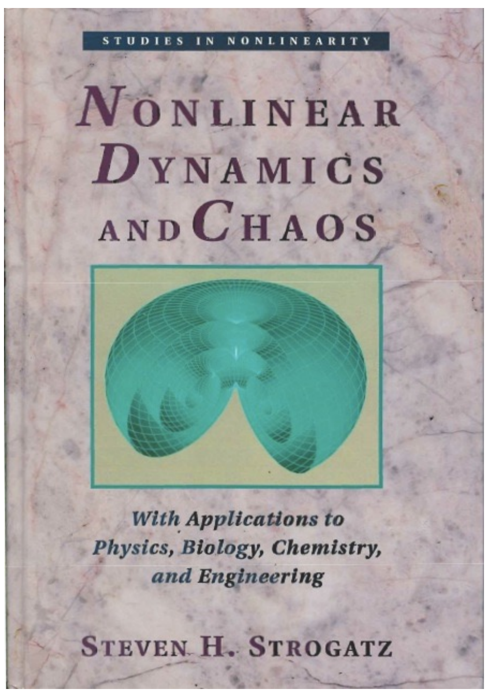
Rabbit

Steven Strogatz

Sheep

6.4 Rabbits versus Sheep

In the next few sections we'll consider some simple examples of phase plane analysis. We begin with the classic ***Lotka–Volterra model of competition*** between two species, here imagined to be rabbits and sheep. Suppose that both species are competing for the same food supply (grass) and the amount available is limited. Furthermore, ignore all other complications, like predators, seasonal effects, and other sources of food. Then there are two main effects we should consider:



Rabbit

Steven Strogatz

Sheep

6.4 Rabbits versus Sheep

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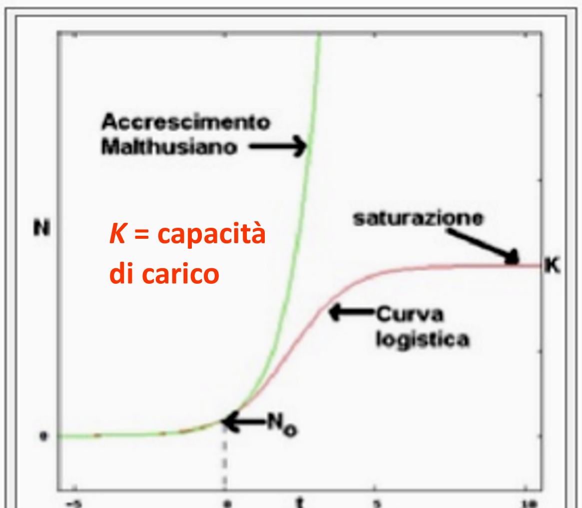

Confronto tra curva logistica e curva di accrescimento esponenziale (malthusiano). I parametri sono: $k = 10, N_0 = 1, r = 1$

1. Each species would grow to its carrying capacity in the absence of the other. This can be modeled by assuming logistic growth for each species (recall Section 2.3). Rabbits have a legendary ability to reproduce, so perhaps we should assign them a higher intrinsic growth rate.

Steven Strogatz

Sheep

Rabbit

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2. When rabbits and sheep encounter each other, trouble starts. Sometimes the rabbit gets to eat, but more usually the sheep nudges the rabbit aside and starts nibbling (on the grass, that is). We'll assume that these conflicts occur at a rate proportional to the size of each population. (If there were twice as many sheep, the odds of a rabbit encountering a sheep would be twice as great.) Furthermore, we assume that the conflicts reduce the growth rate for each species, but the effect is more severe for the rabbits.

$$x = 0 \rightarrow \dot{x} = 0$$

$$\dot{y} = 2y(1 - \frac{y}{2})$$

$$[b = 1, a = K = 2]$$

A specific model that incorporates these assumptions is

$$\begin{cases} \dot{x} = x(3 - x - 2y) \\ \dot{y} = y(2 - x - y) \end{cases}$$

where

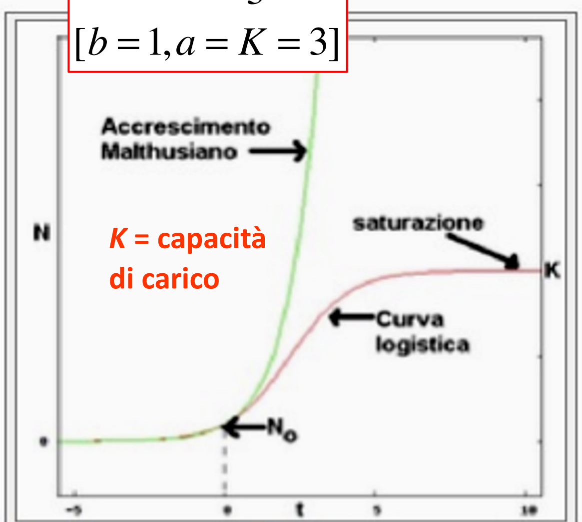
$$x(t) = \text{population of rabbits,}$$
$$y(t) = \text{population of sheep}$$

$$y = 0 \rightarrow \dot{y} = 0$$

$$\dot{x} = 3x(1 - \frac{x}{3})$$

$$[b = 1, a = K = 3]$$

and $x, y \geq 0$. The coefficients have been chosen to reflect this scenario, but are otherwise arbitrary. In the exercises, you'll be asked to study what happens if the coefficients are changed.



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To find the fixed points for the system, we solve $\dot{x} = 0$ and $\dot{y} = 0$ simultaneously. Four fixed points are obtained: $(0,0)$, $(0,2)$, $(3,0)$, and $(1,1)$.

Solutions:

$x = 0, \quad y = 2$

$x = 1, \quad y = 1$

$x = 3, \quad y = 0$

$y = 0, \quad x = 0$

$x = 0 \rightarrow \dot{x} = 0$

$\dot{y} = 2y(1 - \frac{y}{2})$

$[b = 1, a = K = 2]$

A specific model that incorporates these assumptions is

$$\begin{cases} \dot{x} = x(3 - x - 2y) \\ \dot{y} = y(2 - x - y) \end{cases}$$

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To find the fixed points for the system, we solve $\dot{x} = 0$ and $\dot{y} = 0$ simultaneously. Four fixed points are obtained: $(0,0)$, $(0,2)$, $(3,0)$, and $(1,1)$. To classify them, we compute the Jacobian:

$$J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix}.$$

Solutions:

$x = 0, \quad y = 2$

$x = 1, \quad y = 1$

$x = 3, \quad y = 0$

$y = 0, \quad x = 0$

Now consider the four fixed points in turn:

$$(0,0): \text{ Then } J = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

The eigenvalues are $\lambda = 3, 2$ so $(0,0)$ is an *unstable node*. Trajectories leave the origin parallel to the eigenvector for $\lambda = 2$, i.e. tangential to $\mathbf{v} = (0,1)$, which spans the y-axis. (Recall the general rule: at a node, trajectories are tangential to the slow eigendirection, which is the eigendirection with the smallest $|\lambda|$.) Thus, the phase portrait near $(0,0)$ looks like Figure 6.4.1.

Eigenvalues:	Eigenvectors:
$\lambda_1 = 3$	$v_1 = (1, 0)$
$\lambda_2 = 2$	$v_2 = (0, 1)$

$$\Delta = 6 > 0$$
$$TrJ = 5 > 0$$
$$(TrJ)^2 - 4\Delta = 1 > 0$$

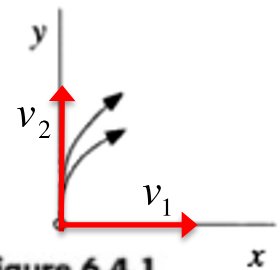
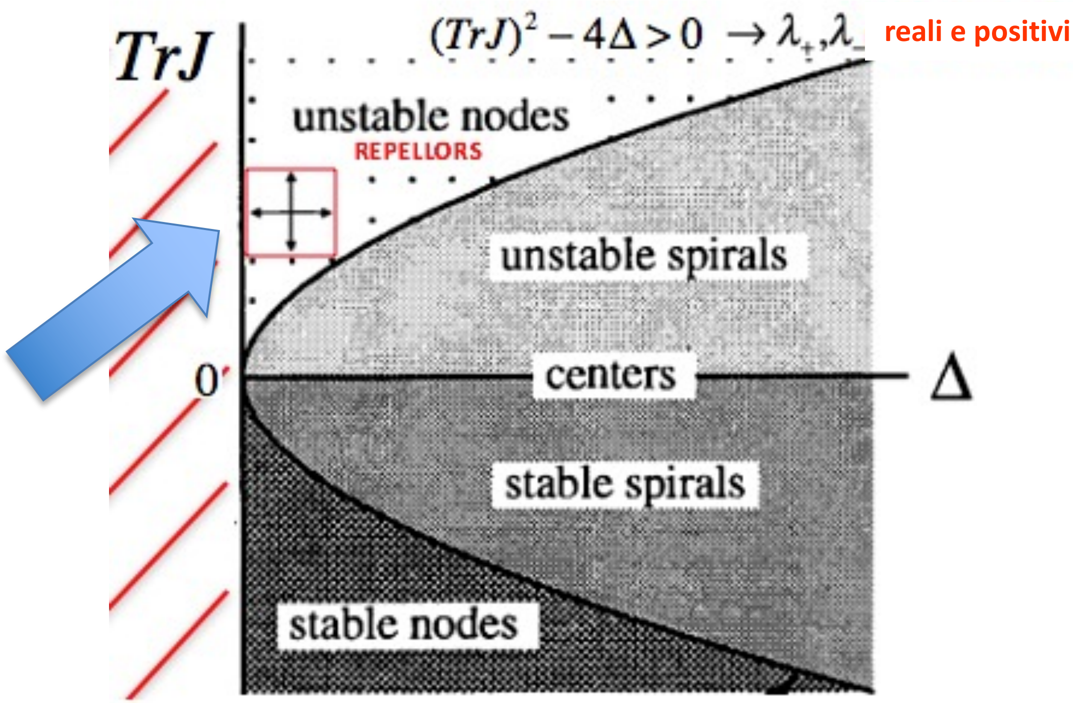


Figure 6.4.1



Eigenvalues:	Eigenvectors:
$\lambda_1 = -2$	$v_1 = (0, 1)$
$\lambda_2 = -1$	$v_2 = (-1, 2)$

$(0,2)$: Then $J = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$.

$$\Delta = 2 > 0$$
$$TrJ = -3 < 0$$
$$(TrJ)^2 - 4\Delta = 1 > 0$$

This matrix has eigenvalues $\lambda = -1, -2$, as can be seen from inspection, since the matrix is triangular. Hence the fixed point is a *stable node*. Trajectories approach along the eigendirection associated with $\lambda = -1$; you can check that this direction is spanned by $\mathbf{v} = (-1, 2)$. Figure 6.4.2 shows the phase portrait near the fixed point $(0,2)$.

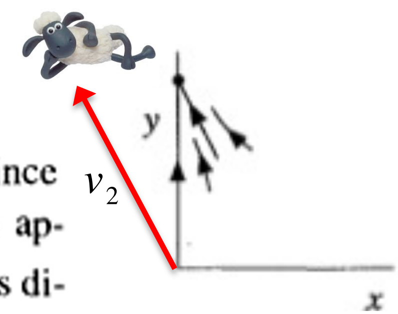
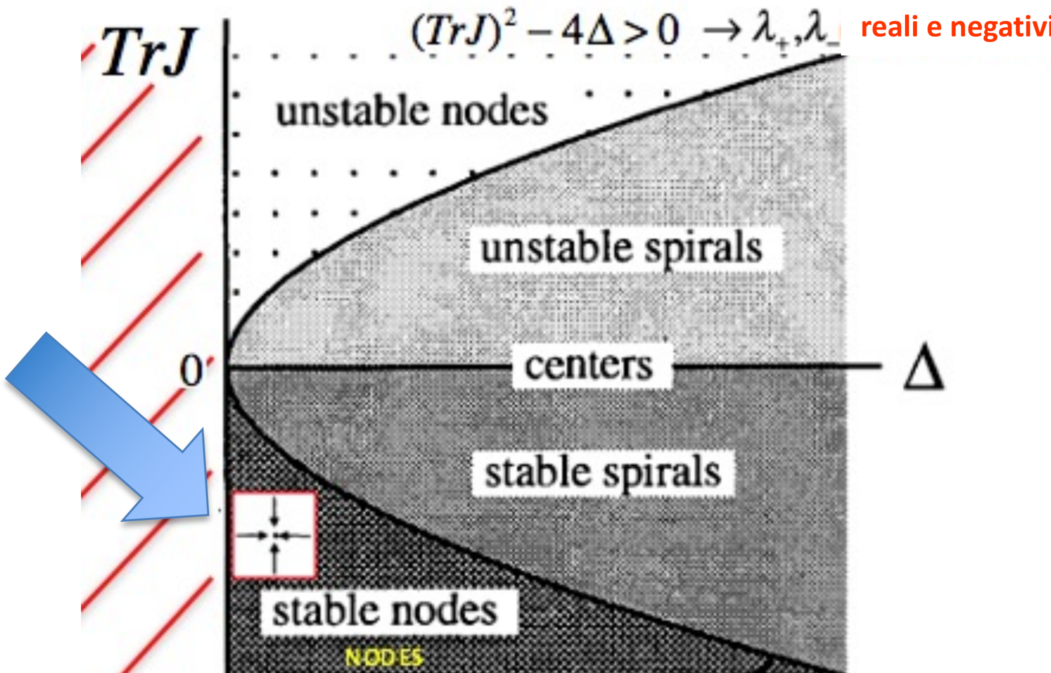


Figure 6.4.2



$(3,0):$ Then $J = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$ and $\lambda = -3, -1$.

This is also a *stable node*. The trajectories approach along the slow eigendirection spanned by $\mathbf{v} = (3,-1)$, as shown in Figure 6.4.3.

$$\Delta = 3 > 0$$
$$TrJ = -4 < 0$$
$$(TrJ)^2 - 4\Delta = 4 > 0$$

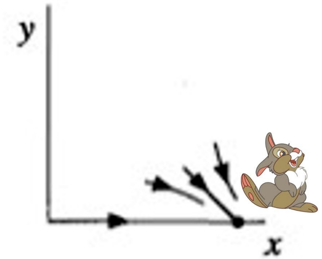


Figure 6.4.3

Eigenvalues:

$\lambda_1 = -3$

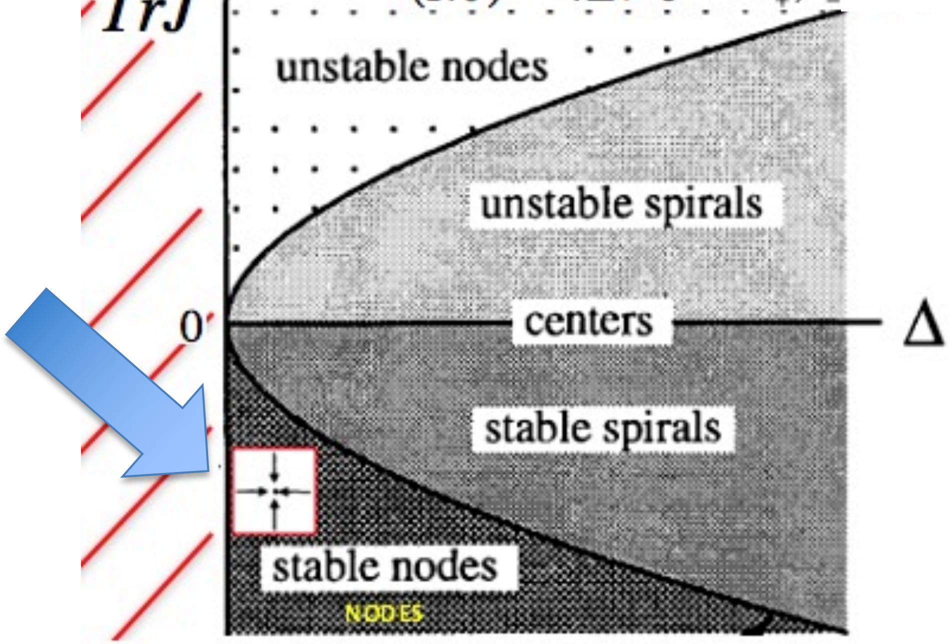
$\lambda_2 = -1$

Eigenvectors:

$v_1 = (1, 0)$

$v_2 = (-3, 1)$

$(TrJ)^2 - 4\Delta > 0 \rightarrow \lambda_+, \lambda_-$ reali e negativi

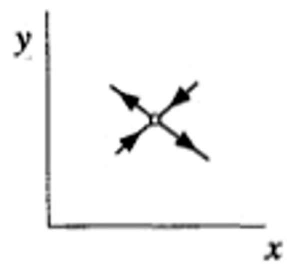


v_2

(1,1): Then $J=\begin{pmatrix}-1 & -2\\ -1 & -1\end{pmatrix}$, which has $\tau=-2$, $\Delta=-1$, and $\lambda=-1\pm\sqrt{2}$.

Hence this is a *saddle point*. As you can check, the phase portrait near (1,1) is as shown in Figure 6.4.4.

$$\Delta=-1<0$$
$$TrJ=-2<0$$



Eigenvalues:

$\lambda_1 \approx -2.41421$

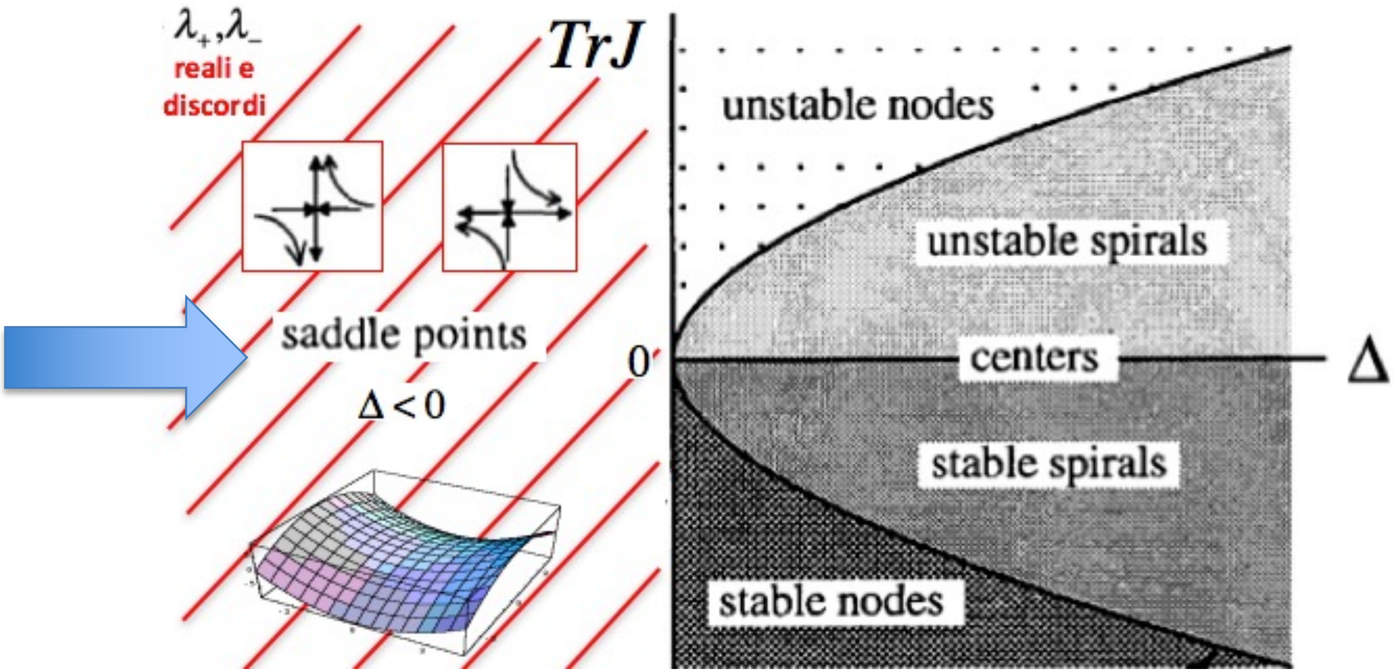
$\lambda_2 \approx 0.414214$

Eigenvectors:

$v_1 = (\sqrt{2}, 1)$

$v_2 = (-\sqrt{2}, 1)$

Figure 6.4.4



Combining Figures 6.4.1–6.4.4, we get Figure 6.4.5, which already conveys a good sense of the entire phase portrait. Furthermore, notice that the x and y axes contain straight-line trajectories, since $\dot{x}=0$ when $x=0$, and $\dot{y}=0$ when $y=0$.

$$\begin{cases} \dot{x} = x(3-x-2y) \\ \dot{y} = y(2-x-y) \end{cases}$$

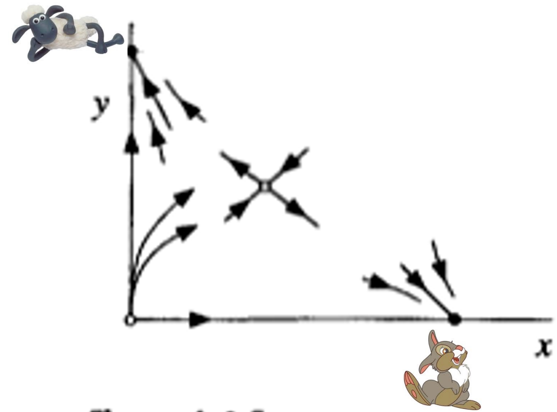
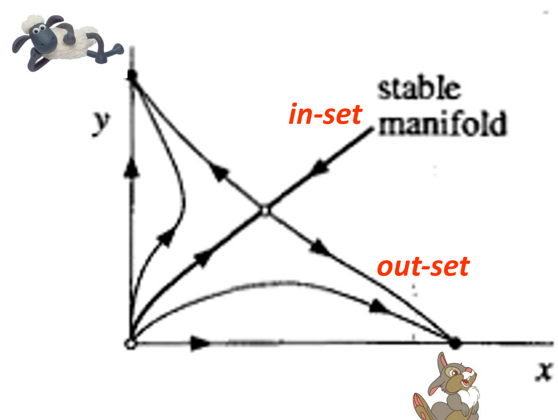


Figure 6.4.5

Now we use common sense to fill in the rest of the phase portrait (Figure 6.4.6). For example, some of the trajectories starting near the origin must go to the stable node on the x-axis, while others must go to the stable node on the y-axis. In between, there must be a special trajectory that can't decide which way to turn, and so it dives into the saddle point. This trajectory is part of the ***stable manifold*** of the saddle, drawn with a heavy line in Figure 6.4.6.



Ritratto globale nello spazio degli stati

The other branch of the stable manifold consists of a trajectory coming in "from infinity." A computer-generated phase portrait (Figure 6.4.7) confirms our sketch.

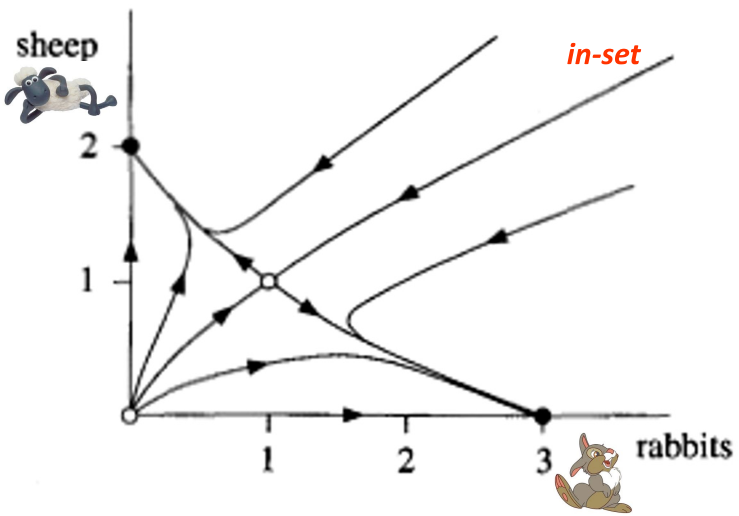
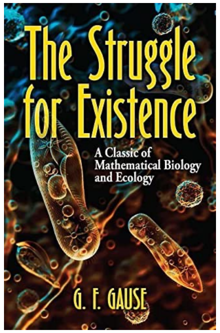


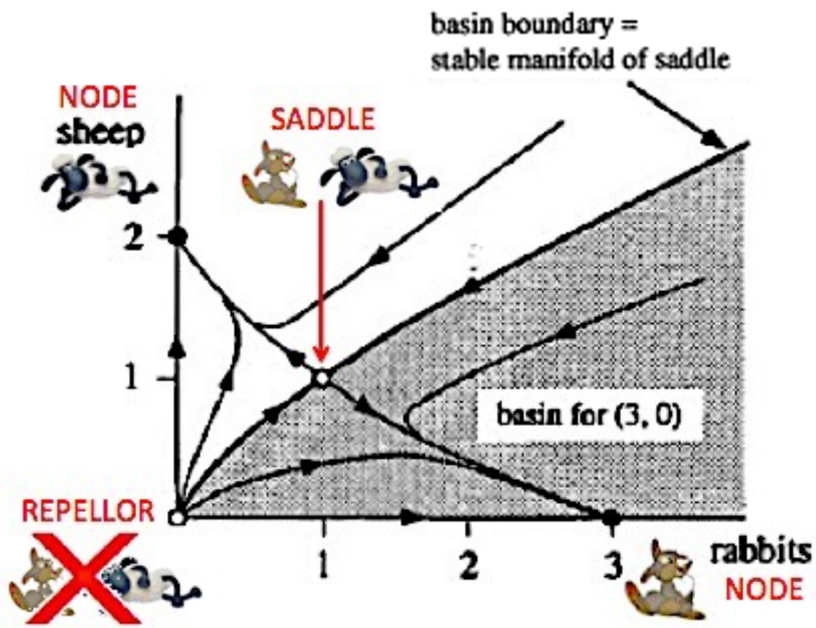
Figure 6.4.7

The phase portrait has an interesting biological interpretation. It shows that one species generally drives the other to extinction. Trajectories starting below the stable manifold lead to eventual extinction of the sheep, while those starting above lead to eventual extinction of the rabbits. This dichotomy occurs in other models of competition and has led biologists to formulate the *principle of competitive exclusion*, which states that two species competing for the same limited resource typically cannot coexist.



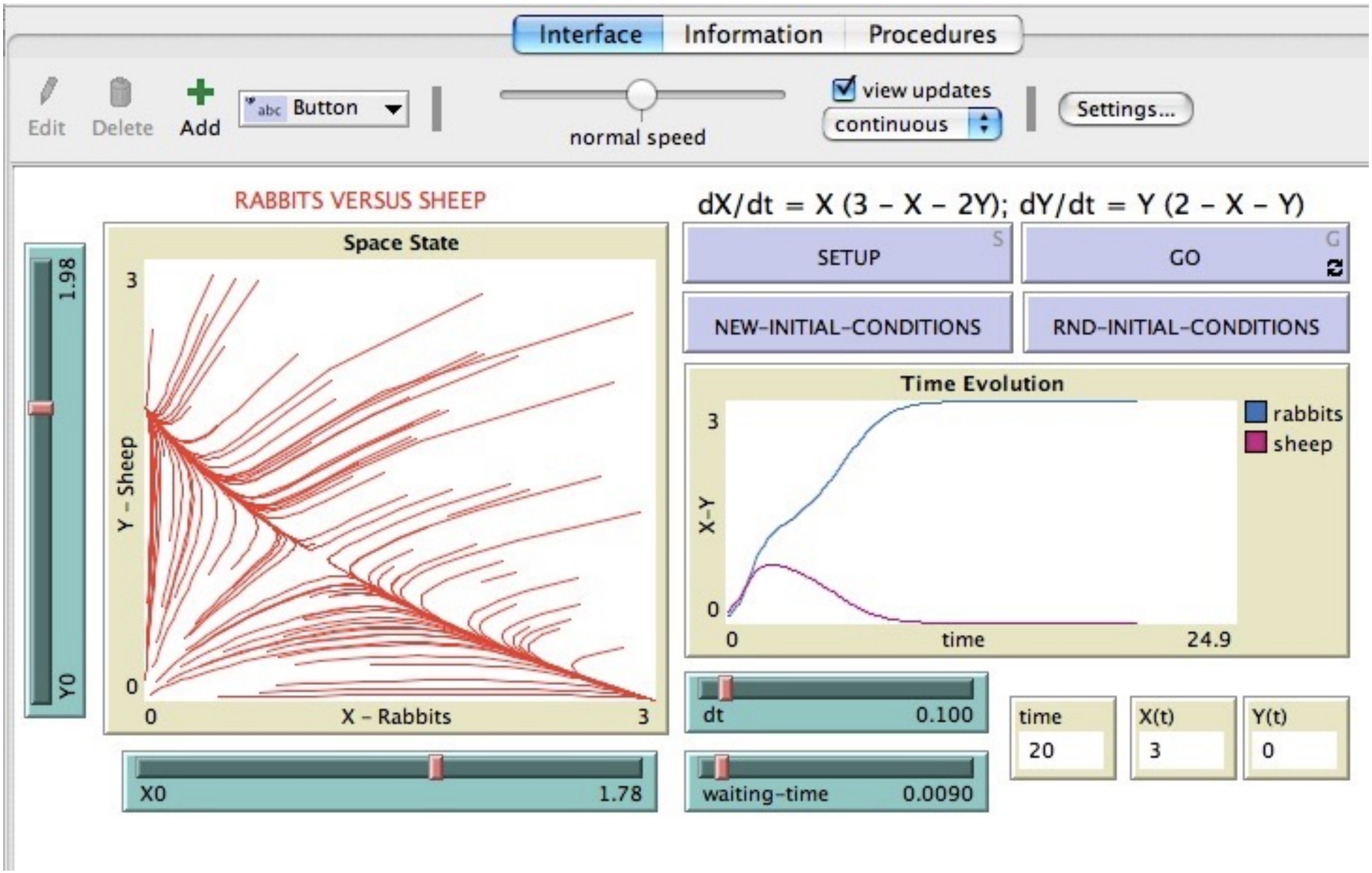
Gause G.F. (1934)
The struggle for existence.
Williams and Wilkins, Baltimore

Our example also illustrates some general mathematical concepts. Given an attracting fixed point $\mathbf{x}^*$, we define its ***basin of attraction*** to be the set of initial conditions $\mathbf{x}_0$ such that $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$. For instance, the basin of attraction for the node at $(3,0)$ consists of all the points lying below the stable manifold of the saddle. This basin is shown as the shaded region in Figure 6.4.8.



Because the stable manifold separates the basins for the two nodes, it is called the ***basin boundary***. For the same reason, the two trajectories that comprise the stable manifold are traditionally called ***separatrices***. Basins and their boundaries are important because they partition the phase space into regions of different long-term behavior.

rabbits_sheep.nlogo



Sistema dinamico con due parametri di controllo e punti fissi con autovalori complessi coniugati

Example: The Brusselator Model

As an illustration of our techniques, let us return to the Brusselator Model given in Eq. (3.11-1).

The Brusselator Model

$$\begin{aligned} \dot{X} &= A-(B+1)X + X^2Y \\ \dot{Y} &= BX - X^2Y \end{aligned} \qquad (3.11\text{-}1)$$

First let us find the fixed points for this set of equations. By setting the time derivatives equal to 0, we find that the fixed points occur at the values X,Y that satisfy

$$A-(B+1)X + X^2Y = 0 \qquad (3.11\text{-}2)$$

$$BX - X^2Y = 0 \qquad (3.11\text{-}3)$$

We see that there is just one point (X,Y) which satisfies these equations, and the coordinates of that fixed point are $X_0 = A$, $Y_0 = B/A$.

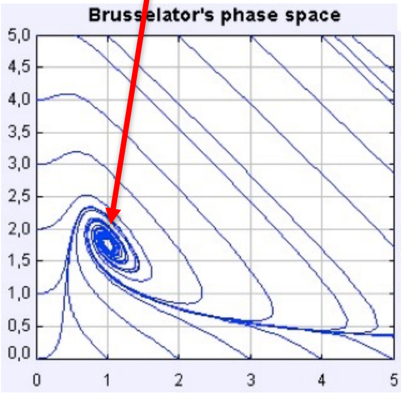
Ilya Prigogine (1917-2003)

Sistema dinamico con due parametri di controllo e punti fissi con autovalori complessi coniugati

$$\dot{X} = A - (B+1)X + X^2 Y$$
$$\dot{Y} = BX - X^2 Y$$

1 punto fisso:

$X_0 = A,\ Y_0 = B/A.$



Ilya Prigogine (1917-2003)

The Jacobian matrix for that set of equations is

$$J = \begin{pmatrix} (B-1) & A^2 \\ -B & -A^2 \end{pmatrix} \qquad \Delta = A^2 \qquad TrJ = (B-1) - A^2 \tag{3.14-7}$$

Following the Jacobian determinant method outlined earlier, we find the characteristic values:

$$\lambda_{\pm} = \frac{TrJ \pm \sqrt{(TrJ)^2 - 4\Delta}}{2} \rightarrow \lambda_{\pm} = \frac{1}{2}\left[(B-1) - A^2\right] \pm \frac{1}{2}\sqrt{\left(A^2 - (B-1)\right)^2 - 4A^2} \tag{3.14-8}$$

$\Delta = 1$

$TrJ = B - 2$

In the discussion of this model, it is traditional to set $A = 1$ and let B be the control parameter. Let us follow that tradition. We see that with $B < 2$, both characteristic values have negative real parts and the fixed point is a spiral node. This result tells us that the chemical concentrations tend toward the fixed point values $X_0 = A = 1$, $Y_0 = B$ as time goes on. They oscillate, however, with the frequency $\Omega = |B(B-4)|^{1/2}$ as they head toward the attractor. For $2 < B < 4$, the fixed point becomes a spiral repellor. However, our analysis cannot tell us what happens to the trajectories as they spiral away from the fixed point. As we shall learn in the next section, they tend to a limit cycle as shown in Fig. I.1 in Section I (for a different model).

Ex: A=1,B=1 → Δ=1, TrJ=-1, $TrJ^2-4\Delta<0$: Spiral Node (B<2)

A=1,B=3 → Δ=1, TrJ= 1, $TrJ^2-4\Delta<0$: Spiral Repellor (B>2)

brussellator.nlogo

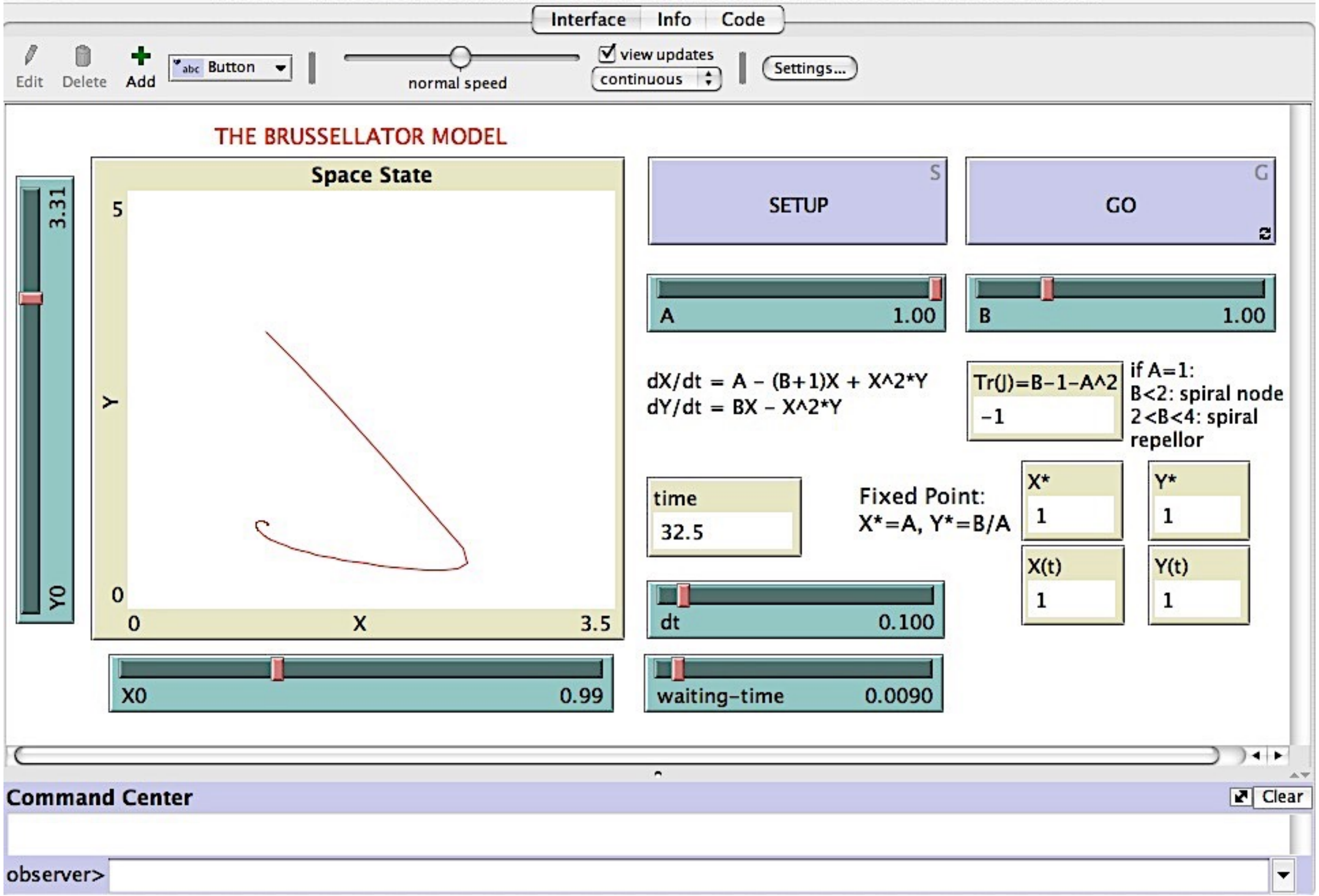
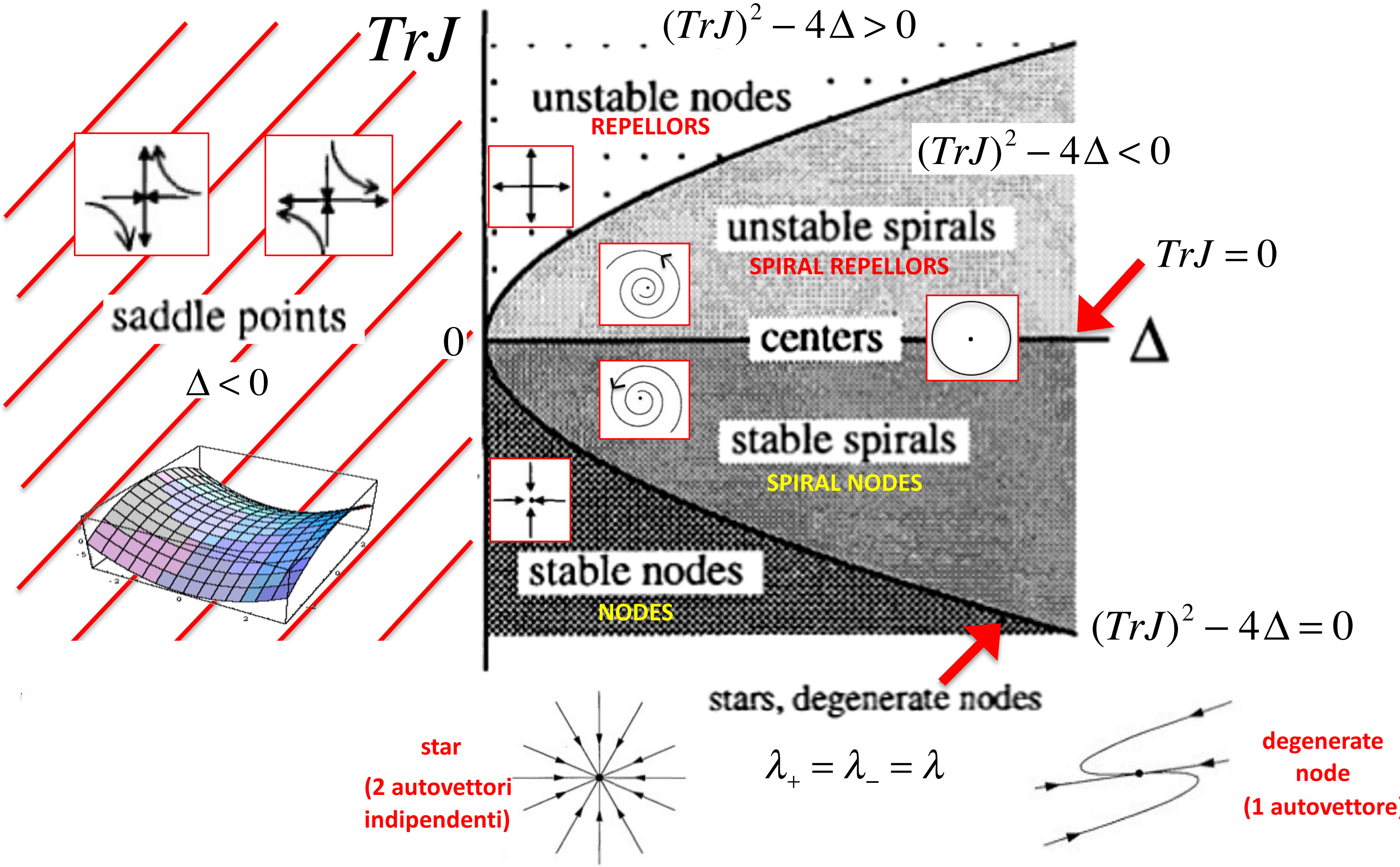


Diagramma dei Punti Fissi in uno Spazio degli Stati a Due Dimensioni

$$\lambda_{\pm} = \frac{TrJ \pm \sqrt{(TrJ)^2 - 4\Delta}}{2}$$



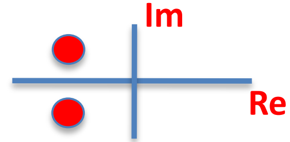
3.15 Limit Cycles

In state spaces with two or more dimensions, it is possible to have cyclic or periodic behavior. This very important kind of behavior is represented by closed loop trajectories in the state space. A trajectory point on one of these loops continues to cycle around that loop for all time. These loops are called ***limit cycles*** if the cycle is isolated, that is if trajectories nearby either approach or are repelled from the limit cycle. The discussion in the previous section indicated that motion on a limit cycle in state space represents oscillatory, repeating motion of the system. The oscillatory behavior is of crucial importance in many practical applications, ranging from radios to brain waves.

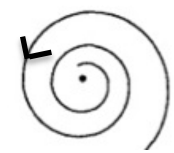
$$(TrJ)^2 - 4\Delta < 0 \quad \rightarrow \lambda_+, \lambda_-$$

complessi coniugati

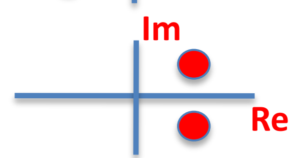
$TrJ < 0$



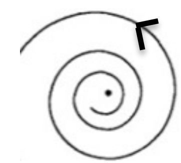
SPIRAL NODE



$TrJ > 0$

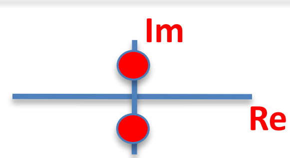


SPIRAL REPELLOR

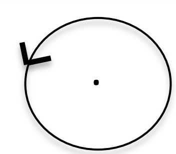


$$x_1(t) = F_1 e^{Rt} \sin \Omega t$$
$$x_2(t) = F_2 e^{Rt} \sin \Omega t$$

$R = TrJ = 0$



NASCE UN CICLO LIMITE

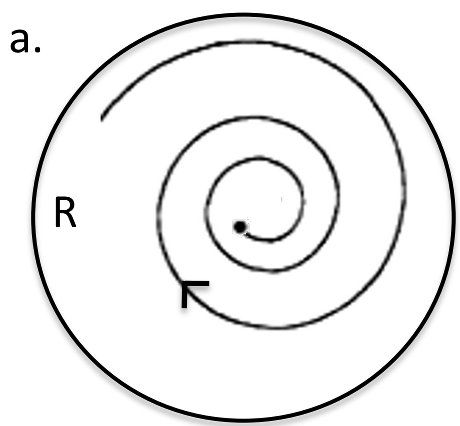


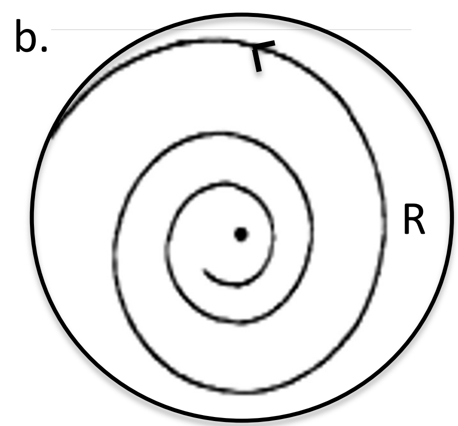
Il Teorema di Poincaré-Bendixson

We shall formulate the analysis in answer to two questions: (1) When do limit cycles occur? and (2) When is a limit cycle stable or unstable? The first question is answered for a two-dimension state space by the famous ***Poincaré–Bendixson Theorem***. The theorem can be formulated in the following way:

1. Suppose the long-term motion of a state point in a two-dimensional state space is limited to some finite-size region; that is, the system doesn't wander off to infinity.
2. Suppose that this region (call it R) is such that any trajectory starting within R stays within R for all time. [R is called an "invariant set" for that system.]
3. Consider a particular trajectory starting in R. The Poincaré–Bendixson Theorem states that there are only two possibilities for that trajectory:

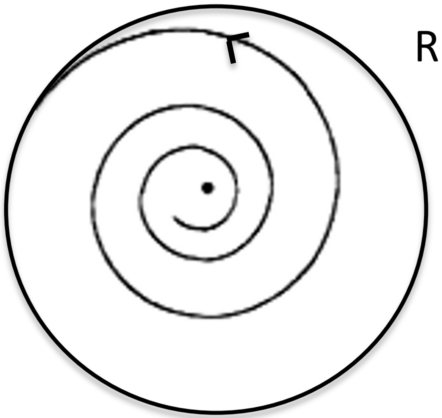
   a. The trajectory approaches a fixed point of the system as $t \to \infty$.
   b. The trajectory approaches a limit cycle as $t \to \infty$.





Il Teorema di Poincaré-Bendixson

A proof of this theorem is beyond the scope of this book. The interested reader is referred to [Hirsch and Smale, 1974]. We can see, however, that the results are entirely reasonable if we take into account the No-Intersection Theorem and the assumption of a bounded region of state space in which the trajectories live. The reader is urged to draw some pictures of state space trajectories in two dimensions to see that these two principles guarantee that the only two possibilities are fixed points and limit cycles.



It is important to note that the Poincaré–Bendixson Theorem works only in two dimensions because only in two dimensions does a closed curve separate the space into a region "inside" the curve and a region "outside." Thus a trajectory starting inside the limit cycle can never get out and a trajectory starting outside can never get in. This argument is an excellent example of the power of topological arguments in the study of dynamical systems. Further, from the Poincaré–Bendixson Theorem we arrive at an important result: Chaotic trajectories (in a bounded system) cannot occur in a state space of two dimensions. *For systems described by differential equations, we need at least three state-space dimensions for chaos.*

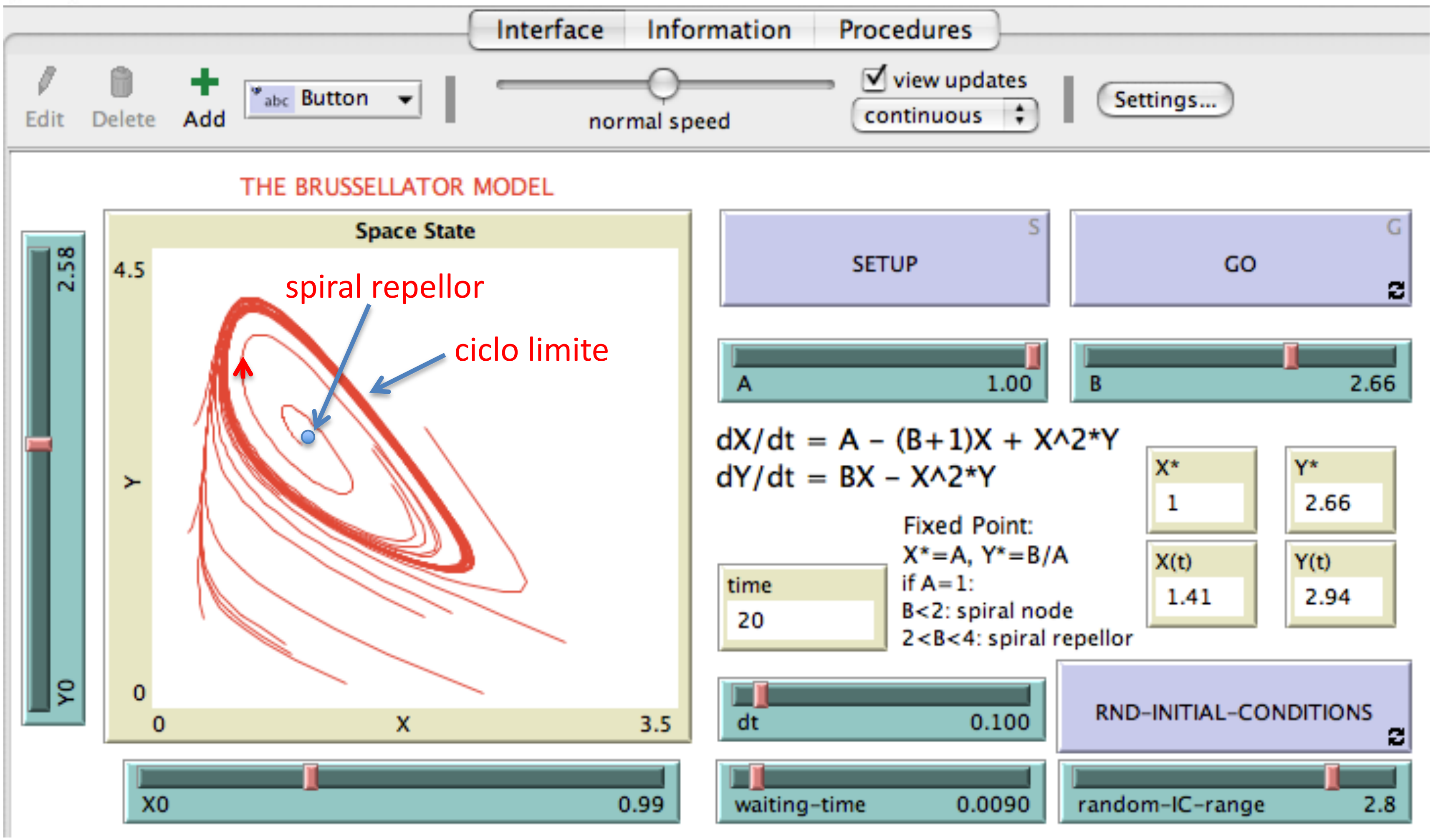
Brussellator-v2.nlogo

The Brussellator model displays the typical situation in which a limit cycle develops. An invariant region R contains a repelling fixed point. Trajectories starting near the repelling fixed point are pushed away and (if there is no attracting fixed point in R) must head toward a limit cycle (which can be proved to enclose the repellor).

$$\dot{X} = A - (B+1)X + X^2Y$$
$$\dot{Y} = BX - X^2Y$$

1 punto fisso:

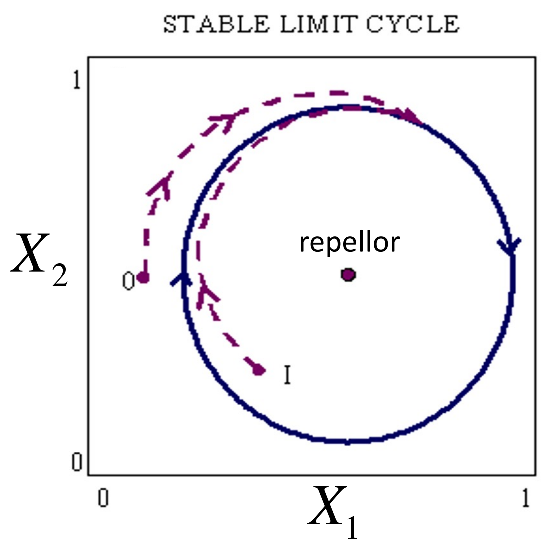
$X_0 = A,\ Y_0 = B/A.$

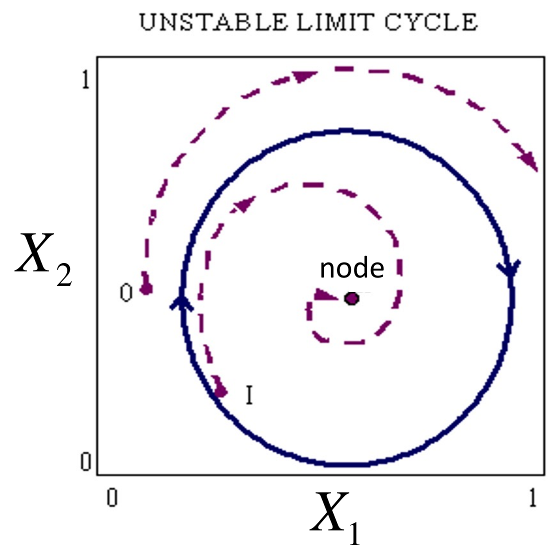


3.16 Poincaré Sections and the Stability of Limit Cycles

We have seen that in state spaces of two (or more) dimensions, a new type of behavior can arise: motion on a limit cycle. The obvious question is the following: Is the motion on the limit cycle stable? That is, if we push the system slightly away from the limit cycle, does it return to the limit cycle (at least asymptotically) or is it repelled from the limit cycle? As we shall see, both possibilities occur in actual systems.

You might expect that we would proceed much as we did for nodes and repellors, by calculating characteristic values involving derivatives of the functions describing the state space evolution. In principle, one could do this, but Poincaré showed that an algebraically and conceptually much simpler method suffices. This method uses what is called a ***Poincaré section*** of the limit cycle. The Poincaré section is closely related to the stroboscopic portraits used in Chapter 1 to discuss the behavior of the diode circuit.





Costruzione della Sezione di Poincaré

For a two-dimensional state space, the Poincaré section is constructed as follows. In the two-dimensional state space, we draw a line segment that cuts through the limit cycle as shown in Fig. 3.12 (a). This line can be any line segment, but in some cases one might wish to choose the X_1 or X_2 axes. Let us call the point at which the limit cycle crosses the line segment going, say, point P.

J.H.Poncaré (1854-1912)

(a)

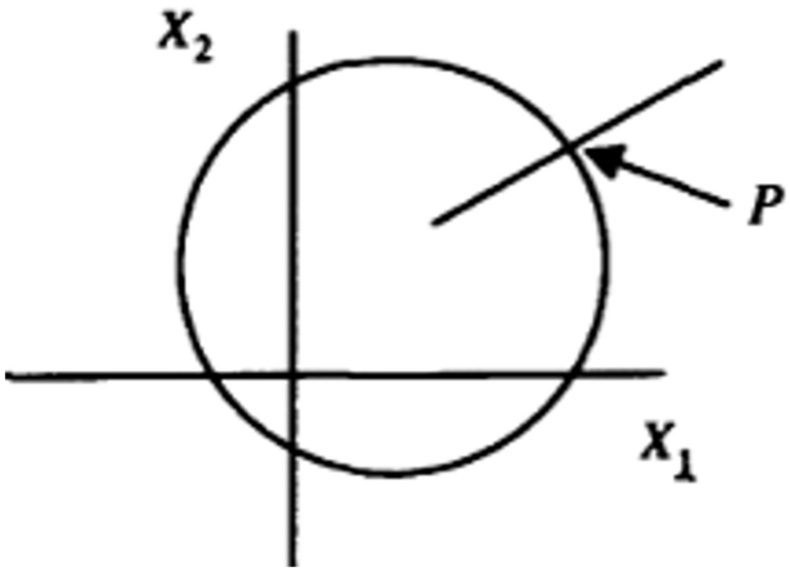


Fig. 3.12. (a) The Poincaré line segment intersects the limit cycle at point P. (b) The four possibilities for sequences of Poincaré intersection points for trajectories near a limit cycle in two dimensions.

If we now start a trajectory in the state space at a point that is close to, but not on, the limit cycle, then that trajectory will cross the Poincaré section line segment at a point other than P. Let's call the first crossing point P_1. As the trajectory evolves, it will cross the Poincaré line segment again at points P_2, P_3, and so on. If the sequence of points approaches P as time goes on for any starting point in the neighborhood of the limit cycle, we say that we have an ***attracting limit cycle*** or, equivalently, a ***stable limit cycle***. In other words, the limit cycle is an attractor for the system. If the sequence of intersection points moves away from P (for any trajectory starting near the limit cycle), we say we have a ***repelling limit cycle*** or, equivalently, an ***unstable limit cycle***. Another possibility is that the points are attracted on one side and repelled on the other: In that case we say that we have a ***saddle cycle*** (in analogy with a saddle point). These possibilities are shown graphically in Fig. 3.12 (b).

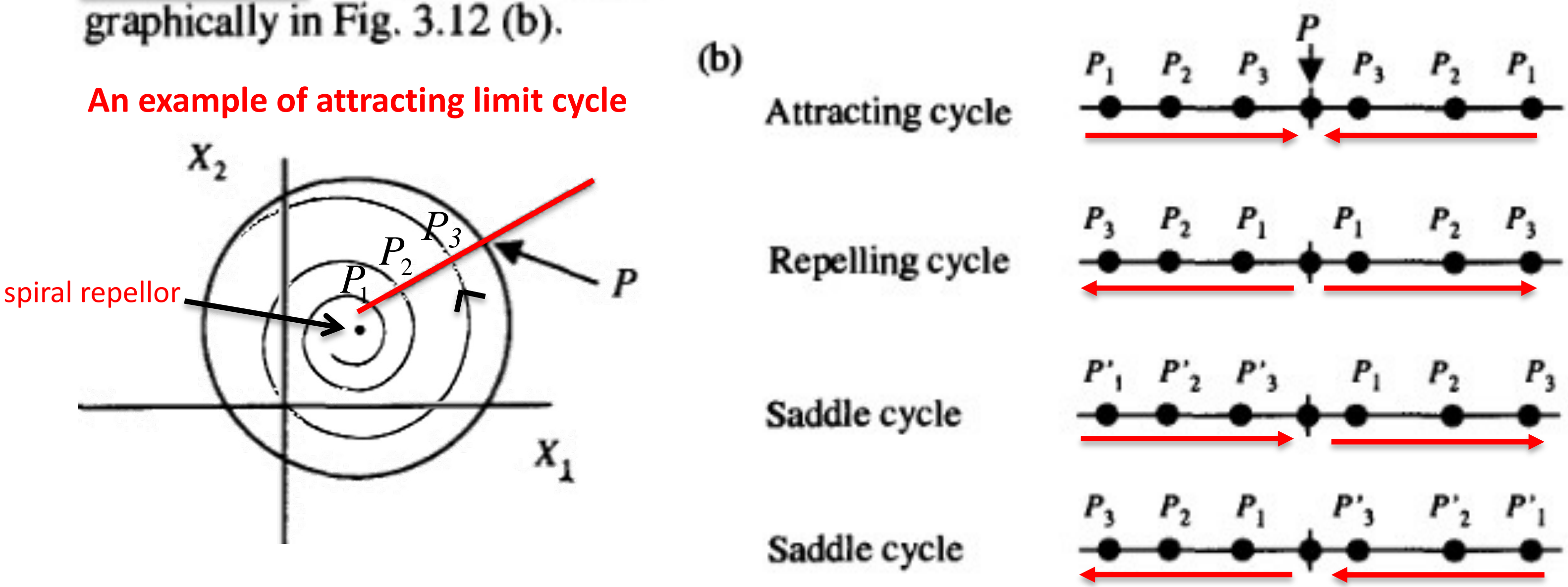


Fig. 3.12. (a) The Poincaré line segment intersects the limit cycle at point P. (b) The four possibilities for sequences of Poincaré intersection points for trajectories near a limit cycle in two dimensions.

How do we describe these properties quantitatively? We use what is called a ***Poincaré map function*** (or ***Poincaré map***, for short). The essential idea is that given a point P_1, where a trajectory crosses the Poincaré line segment, we can in principle determine the next crossing point P_2 by integrating the time-evolution equations describing the system. So, there must be some mathematical function, call it F, that relates P_1 to P_2: $P_2 = F(P_1)$. (Of course, finding this function F is equivalent to solving the original set of equations and that may be difficult or impossible in actual practice.) In general, we may write

$$P_{n+1} = F(P_n) \tag{3.16-1}$$

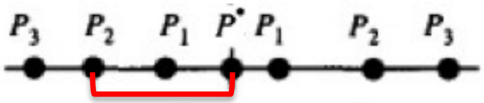
In general the function F depends not only on the original equations describing the system, but on the choice of the Poincaré line segment as well.

To analyze the nature of the limit cycle, we can analyze the nature of the function F and its derivatives. Two points are important to notice:

1. The Poincaré section reduces the original two-dimensional problem to a one-dimensional problem.
2. The Poincaré map function states an iterative (finite-size time step) relation rather than a differential (infinitesimal time step) relation.

The last point is important because F gives P_{n+1} in terms of P_n. The time interval between these points is roughly the time to go around the limit cycle once, a relatively big jump in time. On the other hand, a one-dimensional differential equation $\dot{x} = f(x)$ tells us how x changes over an infinitesimal time interval. The function F is sometimes called an *iterated map function* (or *iterated map*, for short). (Because of the importance of iterated maps in nonlinear dynamics, we shall devote Chapter 5 to a study of their properties.)

Let us note that the point P on the limit cycle satisfies $P = F(P)$. Any point P^* that satisfies $P^* = F(P^*)$ is called a *fixed point* of the map function. If a trajectory crosses the line segment exactly at P^*, it returns to P^* on every cycle. In analogy with our discussion of fixed points for differential equations, we can ask what happens to a point P_1 close to P^*. In particular, we ask what happens to the distance between P_1 and P^* as the system evolves. Formally, we look at



$$P_2 - P^* = F(P_1) - F(P^*) \tag{3.16-2}$$

and use a Taylor series expansion about the point P^* to write

$$P_2 - P^* = F(P^*) + \left.\frac{dF}{dP}\right|_{P^*} (P_1 - P^*) + \ldots - F(P^*) \tag{3.16-3}$$

If we define $d_i = (P_i - P^*)$, we see that

$$d_2 = \left.\frac{dF}{dP}\right|_{P^*} d_1 \tag{3.16-4}$$

We now define the ***characteristic multiplier M*** for the Poincaré map:

$$M = \left.\frac{dF}{dP}\right|_{P^*} \qquad \textbf{(M>0)} \tag{3.16-5}$$

M is also called the ***Floquet multipler*** or the ***Lyapunov multiplier***. In terms of M, we can write Eq. (3.16-4)

$$d_2 = Md_1 \tag{3.16-6}$$

We find in general

$$d_{n+1} = M^n d_1 \tag{3.16-7}$$

$$d_{n+1} = M^n d_1 \quad (3.16\text{-}7)$$

We see that if $M < 1$, then $d_2 < d_1$, $d_3 < d_2$, and so on: The intersection points approach the fixed point P. In that case the cycle is an attracting limit cycle. If $M > 1$, then the distances grow with repeated iterations, and the limit cycle is a repelling cycle. For saddle cycles, M is equal to 1 but the derivative of the map function is greater than 1 on one side of the cycle and less than 1 on the other side. However, based on our discussion of saddle points for one-dimensional state spaces, we expect that saddle cycles are rare in two-dimensional state spaces. Table 3.4 lists the possibilities.

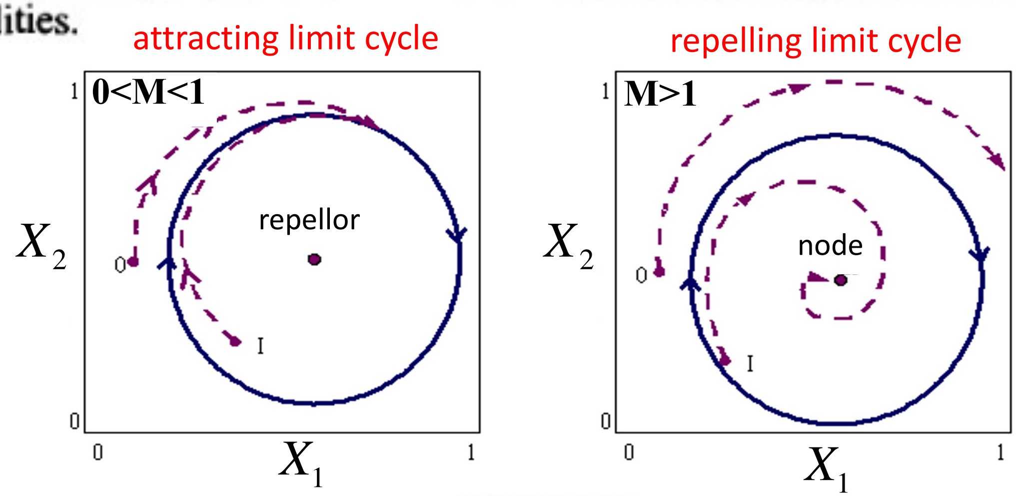


Table 3.4.
The Possible Limit Cycles and Their Characteristic Multipliers for Two–Dimensional State Space

Characteristic Multiplier	Type of Cycle
M < 1	Attracting Cycle
M > 1	Repelling Cycle
M = 1	Saddle Cycle (rare in two-dimensions)

We can also define a ***characteristic exponent*** associated with the cycle by the equation

$$M \equiv e^{\lambda} \tag{3.16-8}$$

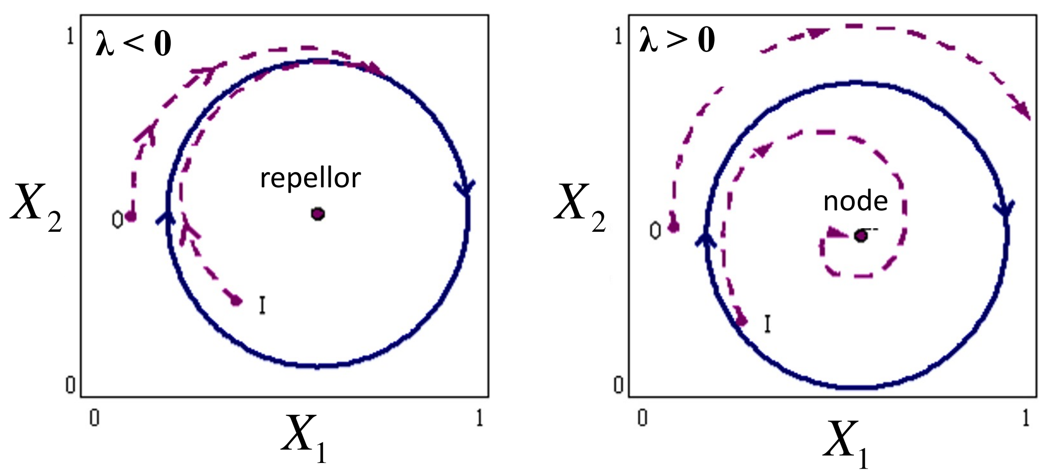
or

$$\lambda \equiv \ln(M) \tag{3.16-9}$$

The idea is that the characteristic exponent plays the role of the Lyapunov exponent but the time unit is taken to be the time from one crossing of the Poincaré section to the next.

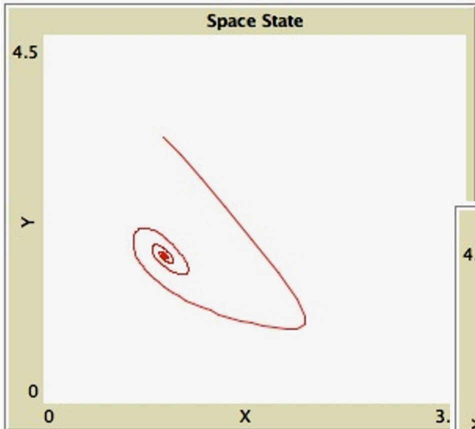
Let us summarize: The Poincaré section method allows us to characterize the possible types of limit cycles and to recognize the kinds of changes that take place in those limit cycles. However, in most cases, we cannot find the mapping function F explicitly; therefore, our ability to predict the kinds of limit cycles that occur for a given system is limited.

attracting limit cycle   repelling limit cycle



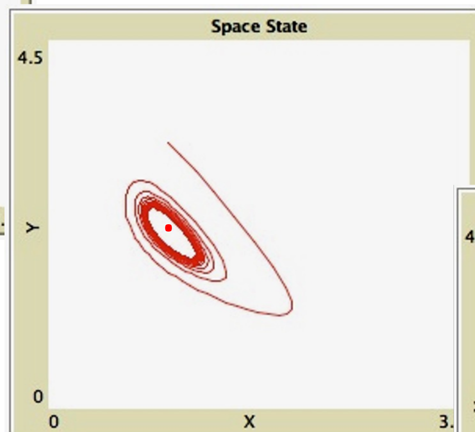
brussellator.nlogo

A=1, B=1.80



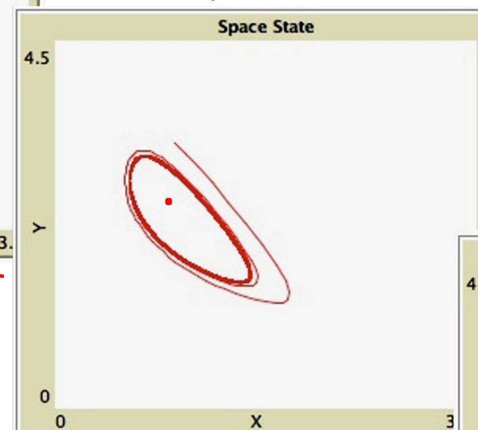
1 stable spiral node

A=1, B=2.15



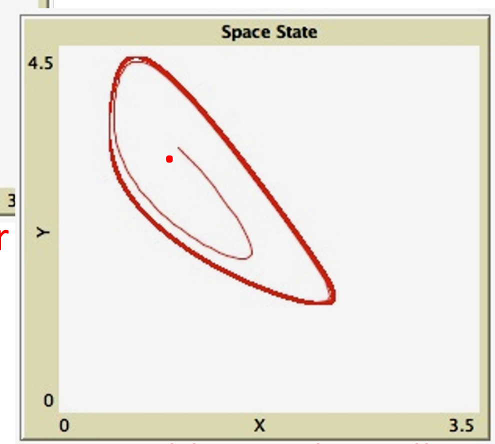
1 unstable spiral repellor
+
1 stable limit cycle

A=1, B=2.33



1 unstable spiral repellor
+
1 stable limit cycle

A=1, B=2.85



1 unstable spiral repellor
+
1 stable limit cycle

$$\dot{X} = A - (B+1)X + X^2 Y$$
$$\dot{Y} = BX - X^2 Y$$

1 punto fisso:

$X_0 = A,\ Y_0 = B/A.$

A=1,B<2 : stable spiral node

A=1, B>2 :
unstable spiral repellor + 1 stable limit cycle

A=1, B=2 : Nasce il ciclo limite!

BIFORCAZIONE