$$\dot{X}_1 = f_1(X_1, X_2)$$
$$\dot{X}_2 = f_2(X_1, X_2)$$

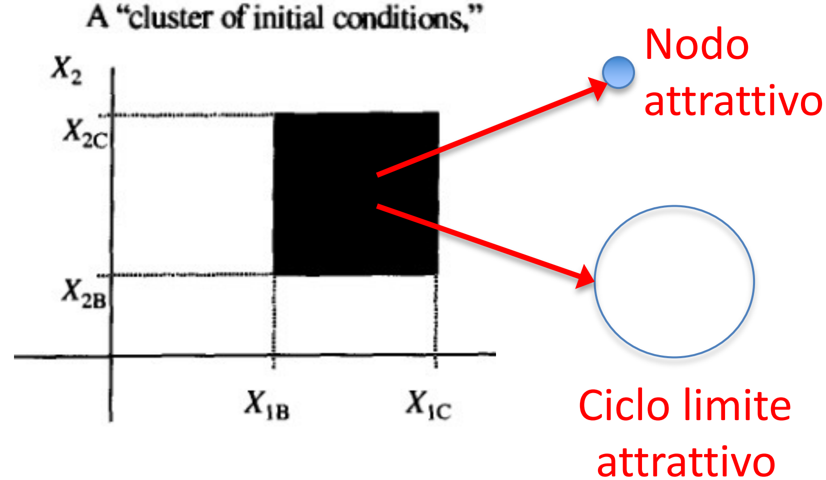
Flussi dissipativi in due dimensioni

$$\frac{1}{A}\frac{dA}{dt} = \frac{\partial f_1}{\partial X_1} + \frac{\partial f_2}{\partial X_2} < 0$$

fixed points (dim.0)

limit cycles (dim.1)

A “cluster of initial conditions,”



Metodo dello Jacobiano per studiare i punti fissi nel caso generale a 2 dim.

Equazioni originarie

$$\dot{X}_1 = f_1(X_1, X_2)$$
$$\dot{X}_2 = f_2(X_1, X_2)$$

...ricavare i punti fissi...

Equazioni linearizzate nelle vicinanze di un dato punto fisso (X_{1o}, X_{2o})

$$\dot{x}_1 = \frac{\partial f_1}{\partial x_1} x_1 + \frac{\partial f_1}{\partial x_2} x_2$$
$$\dot{x}_2 = \frac{\partial f_2}{\partial x_1} x_1 + \frac{\partial f_2}{\partial x_2} x_2$$

with $f_{ij} = \dfrac{\partial f_i}{\partial x_j}$

...calcolate nel punto fisso

Distanze dal punto fisso

$$x_1 = X_1 - X_{1o}$$
$$x_2 = X_2 - X_{2o}$$

3.14 The Jacobian Matrix for Characteristic Values

We would now like to introduce a more elegant and general method of finding the characteristic equation for a fixed point. This method makes use of the so-called ***Jacobian matrix*** of the derivatives of the time evolution functions. Once we see how this procedure works, it will be easy to generalize the method, at least in principle, to find characteristic values for fixed points in state spaces of any dimension. The Jacobian matrix for the system is defined to be the following square array of the derivatives:

Matrice Jacobiana

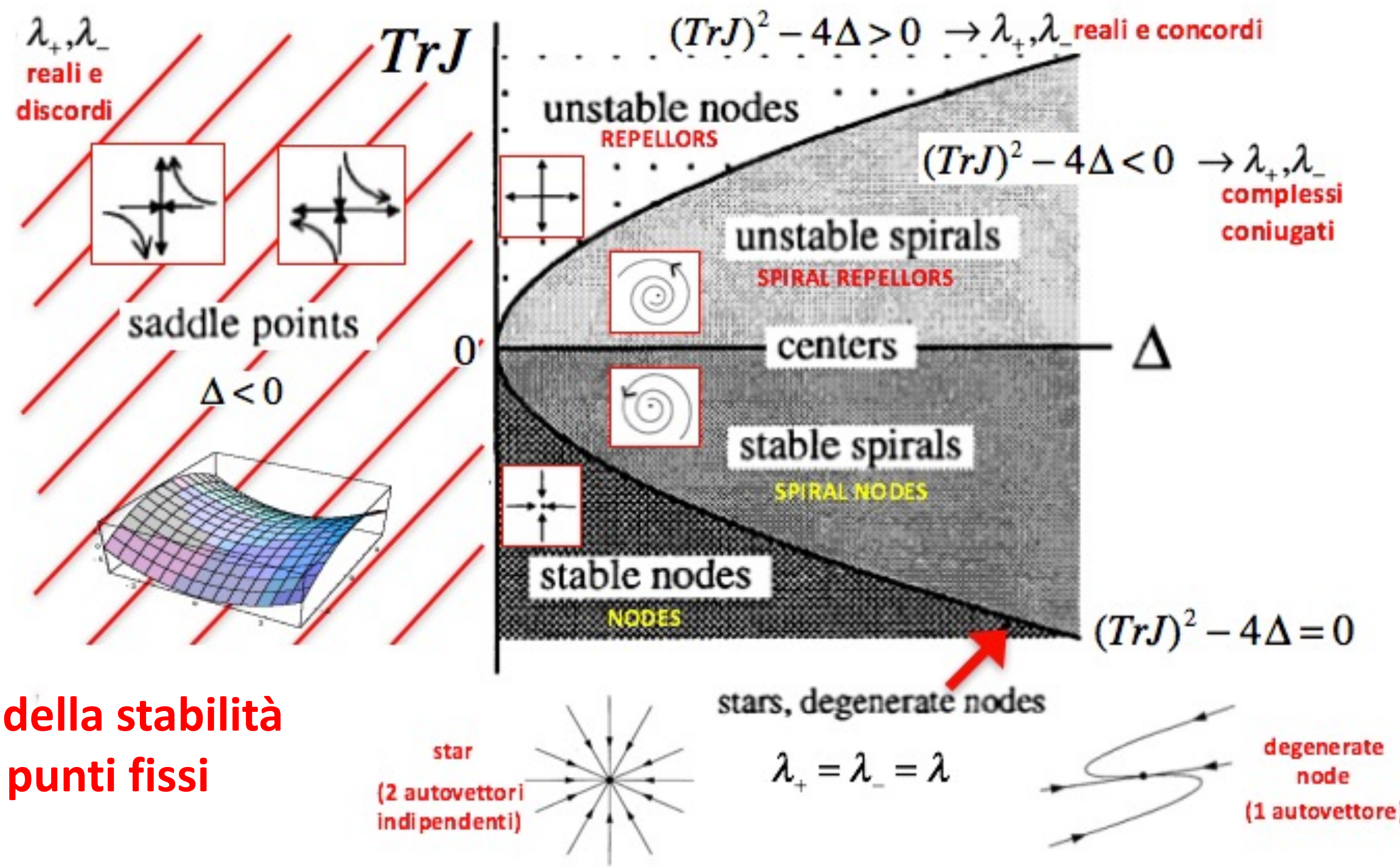
$$J = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \quad (3.14\text{-}1)$$

Autovalori λ_+, λ_-

where the derivatives are evaluated at the fixed point. We subtract λ from each of the principal diagonal (upper left to lower right) elements and set the determinant of the matrix equal to 0:

3.18 Summary

In this chapter we have developed much of the mathematical machinery needed to discuss the behavior of dynamical systems. We have seen that fixed points and their characteristic values (determined by derivatives of the functions describing the dynamics of the system) are crucial for understanding the dynamics. We have also seen that the dimensionality of the state space plays a major role in determining the kinds of trajectories that can occur for bounded systems.



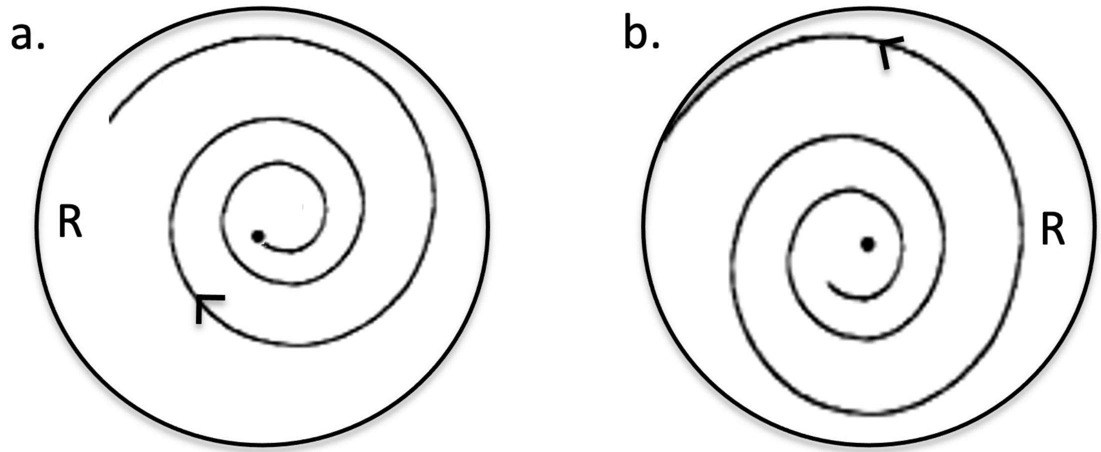
Studio della stabilità dei punti fissi

3.18 Summary

In this chapter we have developed much of the mathematical machinery needed to discuss the behavior of dynamical systems. We have seen that fixed points and their characteristic values (determined by derivatives of the functions describing the dynamics of the system) are crucial for understanding the dynamics. We have also seen that the dimensionality of the state space plays a major role in determining the kinds of trajectories that can occur for bounded systems.

Il Teorema di Poincaré-Bendixson

a. The trajectory approaches a fixed point of the system as $t \to \infty$.
b. The trajectory approaches a limit cycle as $t \to \infty$.



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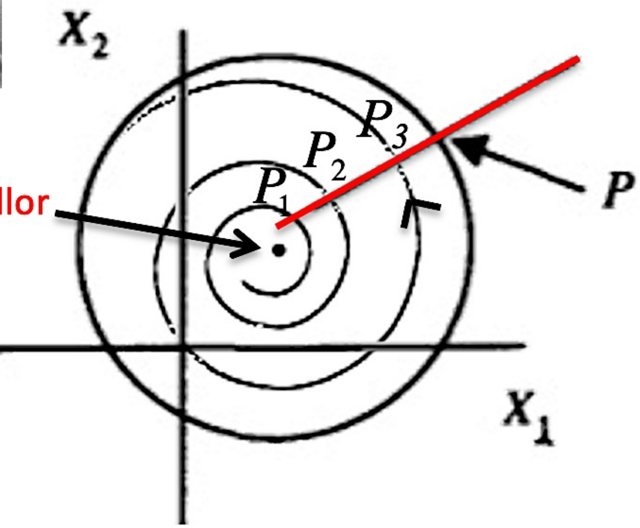
Sezione di Poincaré



Studio della stabilità dei cicli limite

Mappa di Poincaré

$$P_{n+1} = F(P_n)$$

Floquet Multiplier

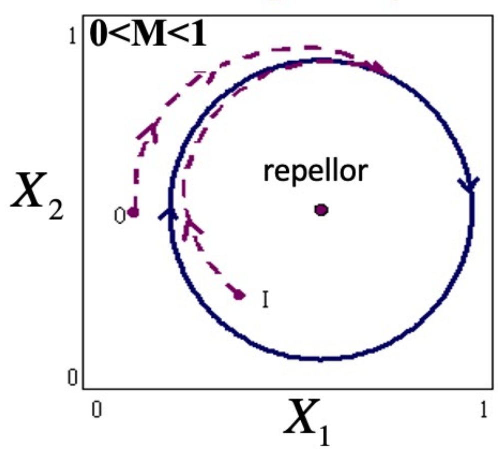
$$M = \left.\frac{dF}{dP}\right|_{P^*}$$

(M>0)

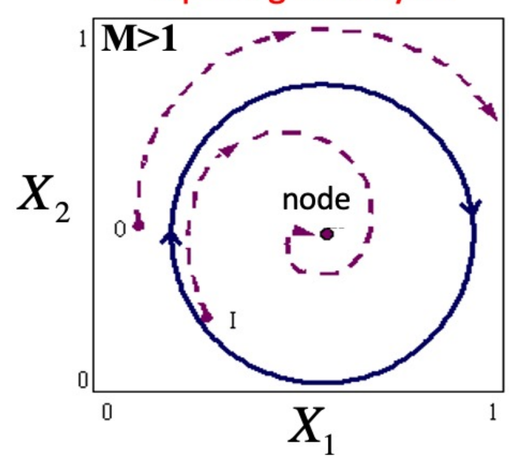
$$d_2 = M d_1$$

$$d_{n+1} = M^n d_1$$

attracting limit cycle



repelling limit cycle



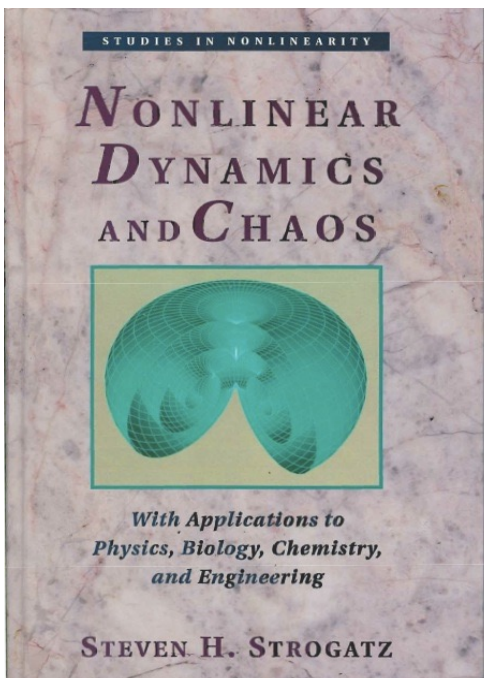
Steven
Strogatz

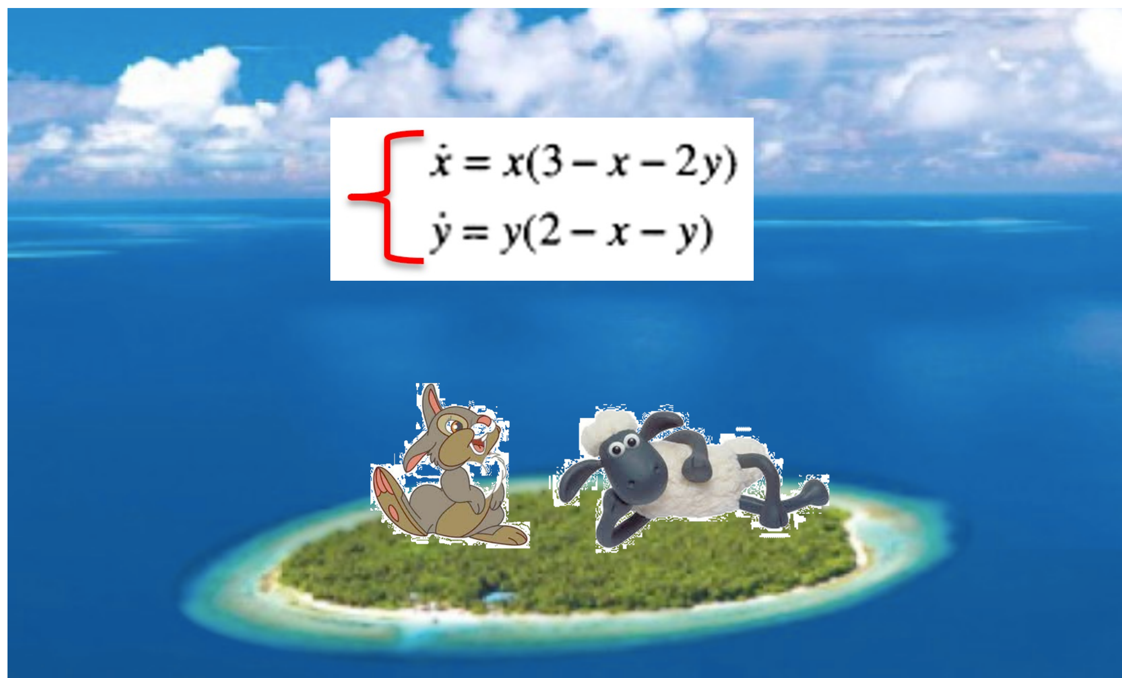
Sheep

Rabbit

6.4 Rabbits versus Sheep

In the next few sections we'll consider some simple examples of phase plane analysis. We begin with the classic ***Lotka–Volterra model of competition*** between two species, here imagined to be rabbits and sheep. Suppose that both species are competing for the same food supply (grass) and the amount available is limited. Furthermore, ignore all other complications, like predators, seasonal effects, and other sources of food. Then there are two main effects we should consider:





$$\begin{cases} \dot{x} = x(3 - x - 2y) \\ \dot{y} = y(2 - x - y) \end{cases}$$

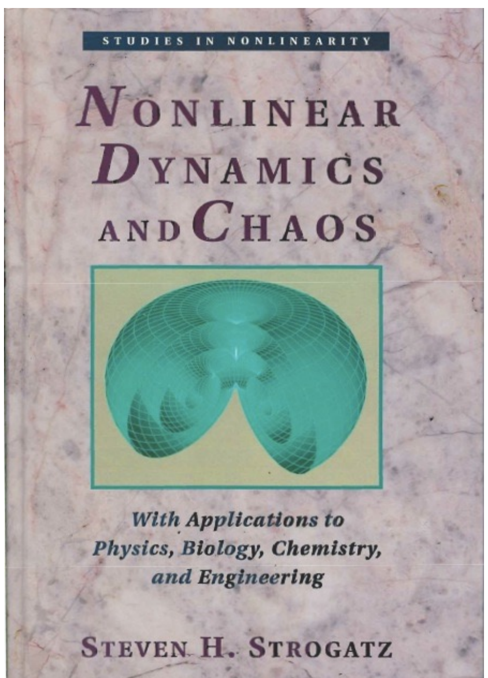
Steven Strogatz

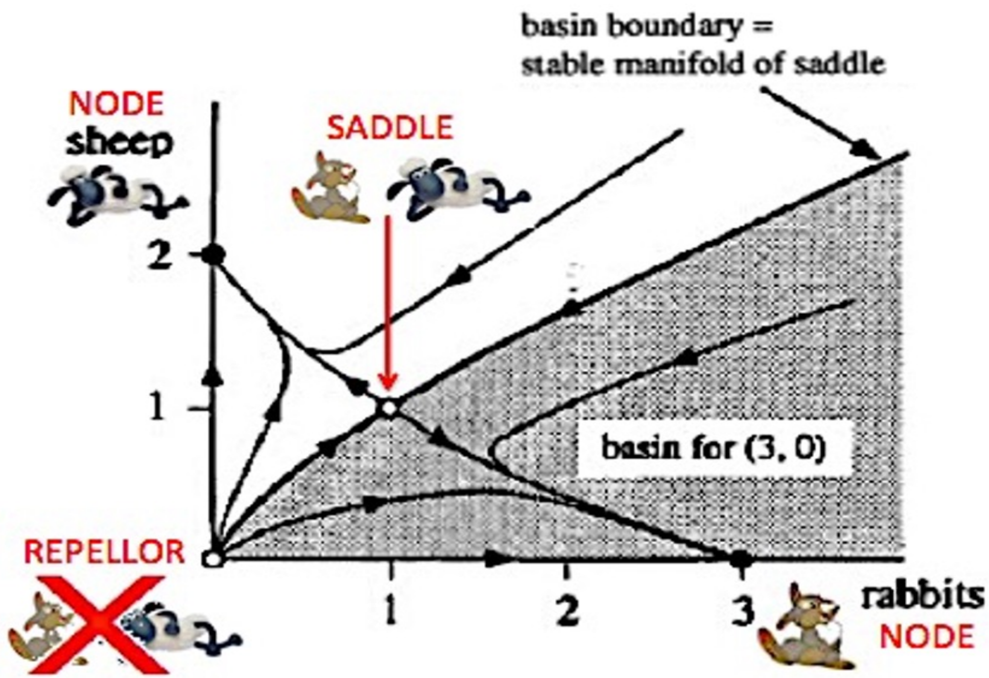
Sheep

Rabbit

6.4 Rabbits versus Sheep

In the next few sections we'll consider some simple examples of phase plane analysis. We begin with the classic ***Lotka–Volterra model of competition*** between two species, here imagined to be rabbits and sheep. Suppose that both species are competing for the same food supply (grass) and the amount available is limited. Furthermore, ignore all other complications, like predators, seasonal effects, and other sources of food. Then there are two main effects we should consider:





Nota: Flussi Non Lineari vs Flussi Lineari

Possono esserci **più punti fissi**

Sistema Non Lineare

Rabbits $\dot{x} = x(3 - x - 2y)$

Sheep $\dot{y} = y(2 - x - y)$

Solutions:

$x = 0, \quad y = 2$

$x = 1, \quad y = 1$

$x = 3, \quad y = 0$

$y = 0, \quad x = 0$

Lo Jacobiano varia per ogni punto fisso e consente di studiare il comportamento della traiettoria solo in **prossimità del punto fisso** (in quanto deriva da espansioni in serie di Taylor).

$$J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix}.$$

Sistema Lineare

Romeo $\dot{x} = ax + by$

Giulietta $\dot{y} = cx + dy$

$$\begin{cases} ax + by = 0, \\ cx + dy = 0. \end{cases}$$

Quando il determinante è diverso da zero, il sistema ha esattamente un **unico punto fisso**, che può essere trovato risolvendo il sistema di equazioni lineari. In questo caso il punto fisso si trova **all'origine** (0,0) poichè non ci sono termini costanti nelle equazioni.

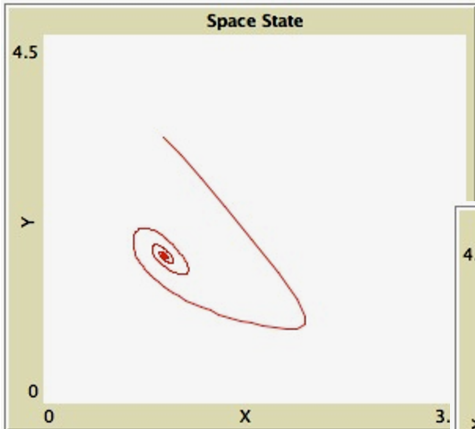
$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Lo Jacobiano descrive il comportamento del sistema anche a **distanze maggiori** dal punto fisso (non richiede alcuna espansione in serie di Taylor).

Nei sistemi dinamici lineari non possono emergere cicli limite. I cicli limite sono una caratteristica dei sistemi dinamici non lineari. Un sistema lineare può avere un comportamento periodico (come nel caso di un oscillatore armonico ideale senza attrito), ma questo comportamento non rappresenta un ciclo limite. **La sensibilità alle condizioni iniziali di un sistema lineare è generalmente bassa**, nel senso che piccole variazioni nelle condizioni iniziali portano a differenze proporzionalmente piccole nel comportamento a lungo termine del sistema. Dunque un sistema lineare, a qualunque numero di dimensioni, **non potrà mostrare comportamenti caotici e neanche biforcazioni** (vedi più avanti...)

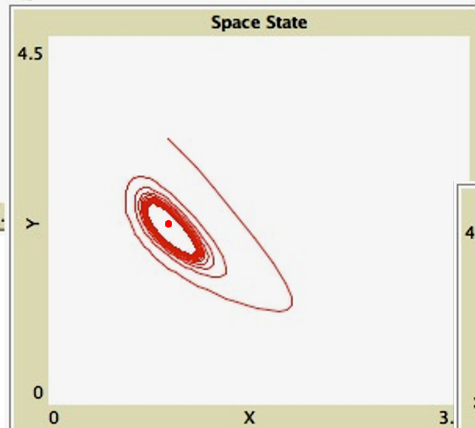
brussellator.nlogo

A=1, B=1.80



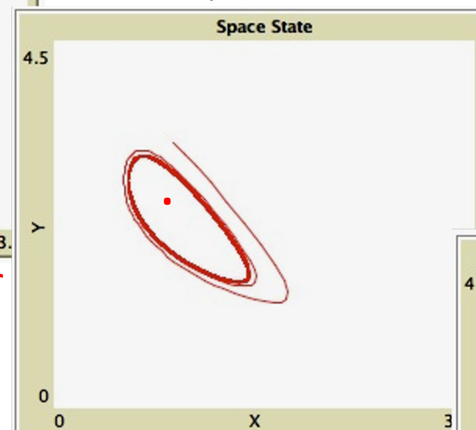
1 stable spiral node

A=1, B=2.15



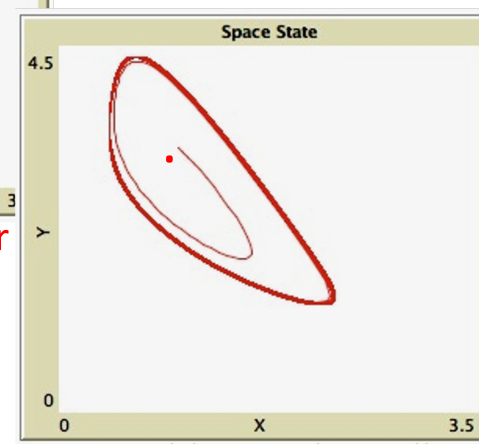
1 unstable spiral repellor + 1 stable limit cycle

A=1, B=2.33



1 unstable spiral repellor + 1 stable limit cycle

A=1, B=2.85



1 unstable spiral repellor + 1 stable limit cycle

$$\dot{X} = A - (B+1)X + X^2Y$$
$$\dot{Y} = BX - X^2Y$$

1 punto fisso:

$$X_0 = A,\ Y_0 = B/A.$$

- A=1,B<2 : stable spiral node
- A=1, B>2 : unstable spiral repellor + 1 stable limit cycle

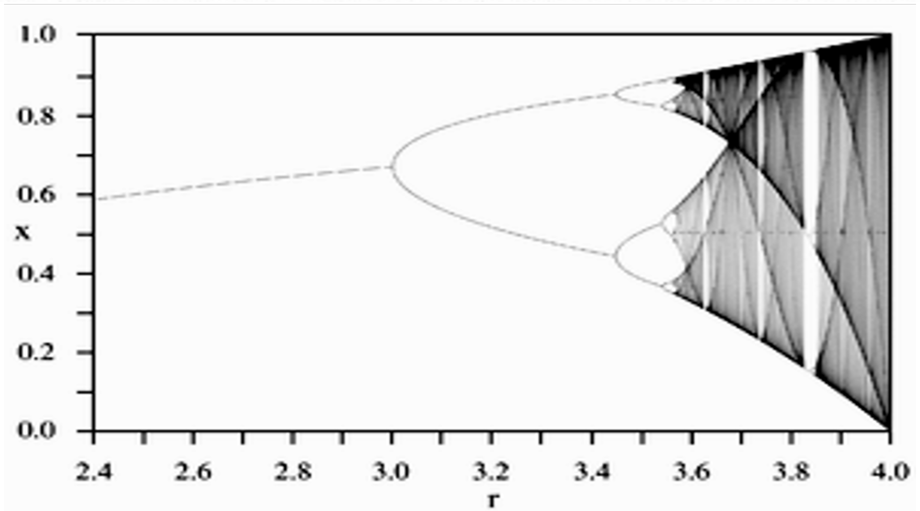
A=1, B=2 : Nasce il ciclo limite!

BIFORCAZIONE

Biforcazioni

3.17 Bifurcation Theory (vale per Flussi e Mappe)

We have seen that the characteristic values associated with a fixed point depend on the various parameters used to describe the system. As the parameters change, for example as we adjust a voltage in a circuit or the concentration of chemicals in a reactor, the nature of the characteristic values and hence the character of the fixed point may change. For example, an attracting node may become a repellor or a saddle point. The study of how the character of fixed points (and other types of state space attractors) change as parameters of the system change is called ***bifurcation theory***. (Recall that the term *bifurcation* is used to describe any sudden change in the dynamics of the system. When a fixed point changes character as parameter values change, the behavior of trajectories in the neighborhood of that fixed point will change. Hence the term bifurcation is appropriate here.) Being able to classify and understand the various possible bifurcations is an important part of the study of nonlinear dynamics. However, the theory, as it is presently developed, is rather limited in its ability to predict the kinds of bifurcations that will occur and the parameter values at which the bifurcations take place for a particular system. Description, however, is the first step toward comprehension and understanding.



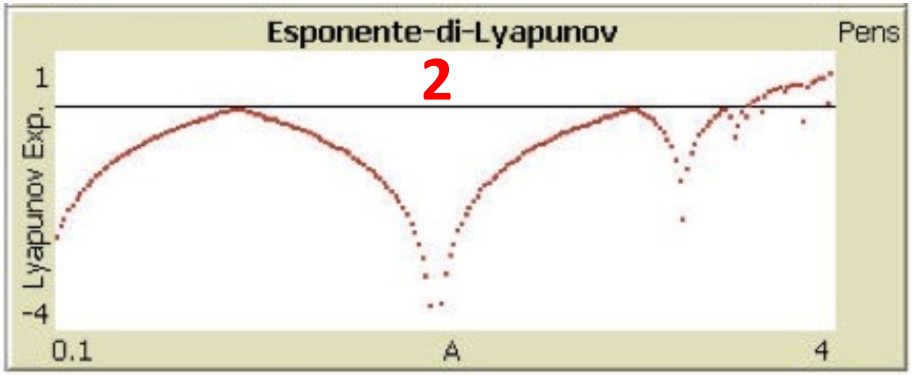
We should also emphasize that simple bifurcation theory treats only the changes in stability of a particular attractor (or, as we shall see in Chapter 4, a particular basin of attraction). Since in general a system may have, for fixed parameter values, several attractors in different parts of state space, we often need to consider the overall dynamical system (that is, its "global" properties) to see what happens to trajectories when a bifurcation occurs.

To keep track of what is happening as the control parameter is varied, we will use two types of diagrams. One type, which we have seen before, is the bifurcation
1 diagram, in which we plot the location of the fixed point (or points) as a function of the control parameter. In the second type of diagram, we plot the characteristic
2 values of the fixed point as a function of the control parameter.

To see how this kind of analysis proceeds, let us begin with the one-dimensional state space case. In a one-dimensional state space, a fixed point has just one characteristic value λ. The crucial assumption in the analysis is that λ varies smoothly (continuously) as some parameter, call it μ, varies. For example, if $\lambda(\mu) < 0$ for some value of μ, then the fixed point is a node. As μ changes, λ might increase (become less negative), going through zero, and then become positive. The node then changes to a repellor when $\lambda > 0$.

Es.Mappa Logistica

$$x_{n+1} = Ax_n(1 - x_n)$$



Biforcazioni in 1D

control parameter

Let us consider a specific example:

Flusso a una dimensione $\dot{x} = \mu - x^2$ (3.17-3)

For μ positive, there are two fixed points: one at $x = +\sqrt{\mu}$, the other at $x = -\sqrt{\mu}$. For μ negative there are no fixed points (assuming, of course, that x is a real number). If we use Eq. (3.6-3), which defines the characteristic value for a fixed point, to find the characteristic value of the two fixed points (for $\mu > 0$), we see that the fixed point at $x = -\sqrt{\mu}$ is a repellor, while the fixed point at $x = +\sqrt{\mu}$ is a node.

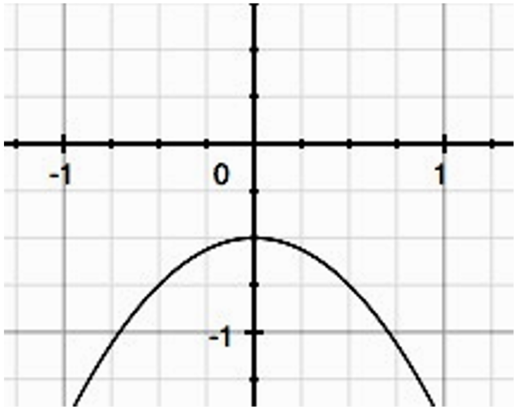
$$\lambda = \left.\frac{df(X)}{dX}\right|_{X=X_o}$$

$$\frac{df(X)}{dX} = -2x$$

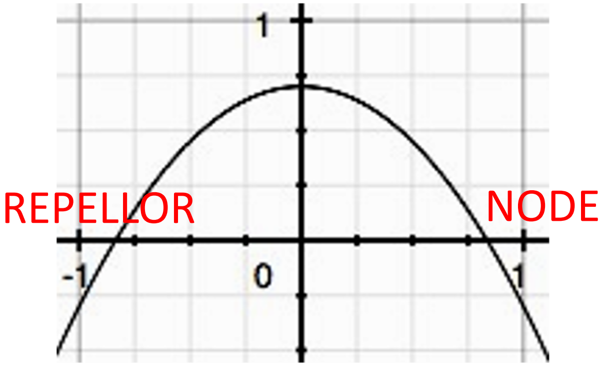
$$\lambda(+\sqrt{\mu}) < 0$$

$$\lambda(-\sqrt{\mu}) > 0$$

$\mu < 0$



$\mu > 0$



Biforcazioni in 1D

Let us consider a specific example:

control parameter

Flusso a una dimensione $$\dot{x} = \mu - x^2 \tag{3.17-3}$$

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If we start with $\mu < 0$ and let it increase, we find that a bifurcation takes place at $\mu = 0$. At that value of the parameter we have a saddle point, which then changes into a repellor-node pair as μ becomes positive. We say that we have a ***repellor-node bifurcation*** at $\mu = 0$.

$$\lambda = \left.\frac{df(X)}{dX}\right|_{X=X_o}$$

$$\frac{df(X)}{dX} = -2x$$

$$\lambda(+\sqrt{\mu}) < 0$$

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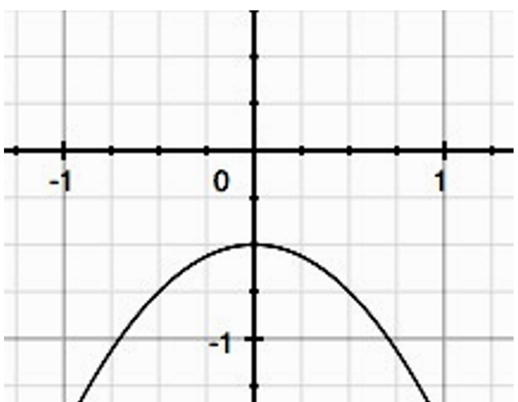
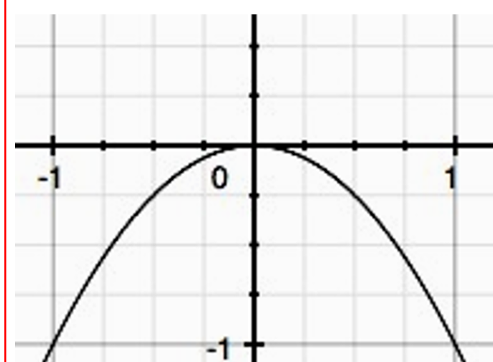
$\mu < 0$

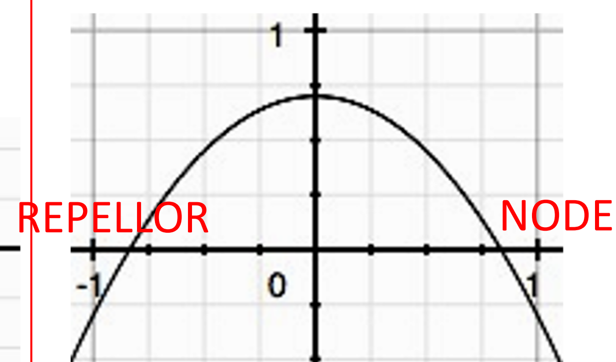


$\mu = 0$

BIFORCAZIONE!



$\mu > 0$



Biforcazioni in 1D

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$$\frac{df(X)}{dX} = -2x$$

$$\lambda(+\sqrt{\mu}) < 0$$

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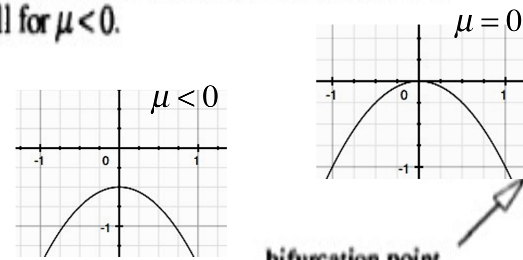
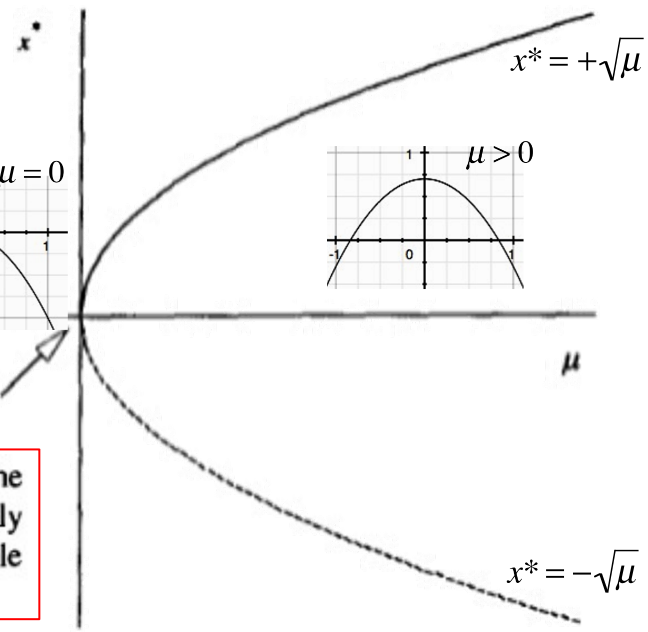
If we start with $\mu < 0$ and let it increase, we find that a bifurcation takes place at $\mu = 0$. At that value of the parameter we have a saddle point, which then changes into a repellor-node pair as μ becomes positive. We say that we have a ***repellor-node bifurcation*** at $\mu = 0$.

Fig. 3.14. The bifurcation diagram for the repellor-node (saddle-node) bifurcation. The solid line indicates the x value for the node as a function of the parameter value. The dashed line is for the repellor. Note that there is no fixed point at all for $\mu < 0$.



bifurcation point



Nota: Note that at the repellor-node bifurcation point, the fixed point of the system is structurally unstable in the sense discussed in Section 3.6. Structurally unstable points are important because their existence indicates a possible bifurcation.

In the nonlinear dynamics literature, the bifurcation just described is usually called a ***saddle-node bifurcation***, ***tangent bifurcation***, or a ***fold bifurcation***. The origin of these names will become apparent when we see analogous bifurcations in higher-dimensional state spaces. For example, if we imagine the curves in Fig. 3.14 as being the cross section of a piece of paper extending into and out of the plane of the page, then the bifurcation point represents a "fold" in the piece of paper. Also, Fig. 3.5 shows how the function in question becomes tangent to the *x* axis at the bifurcation point.

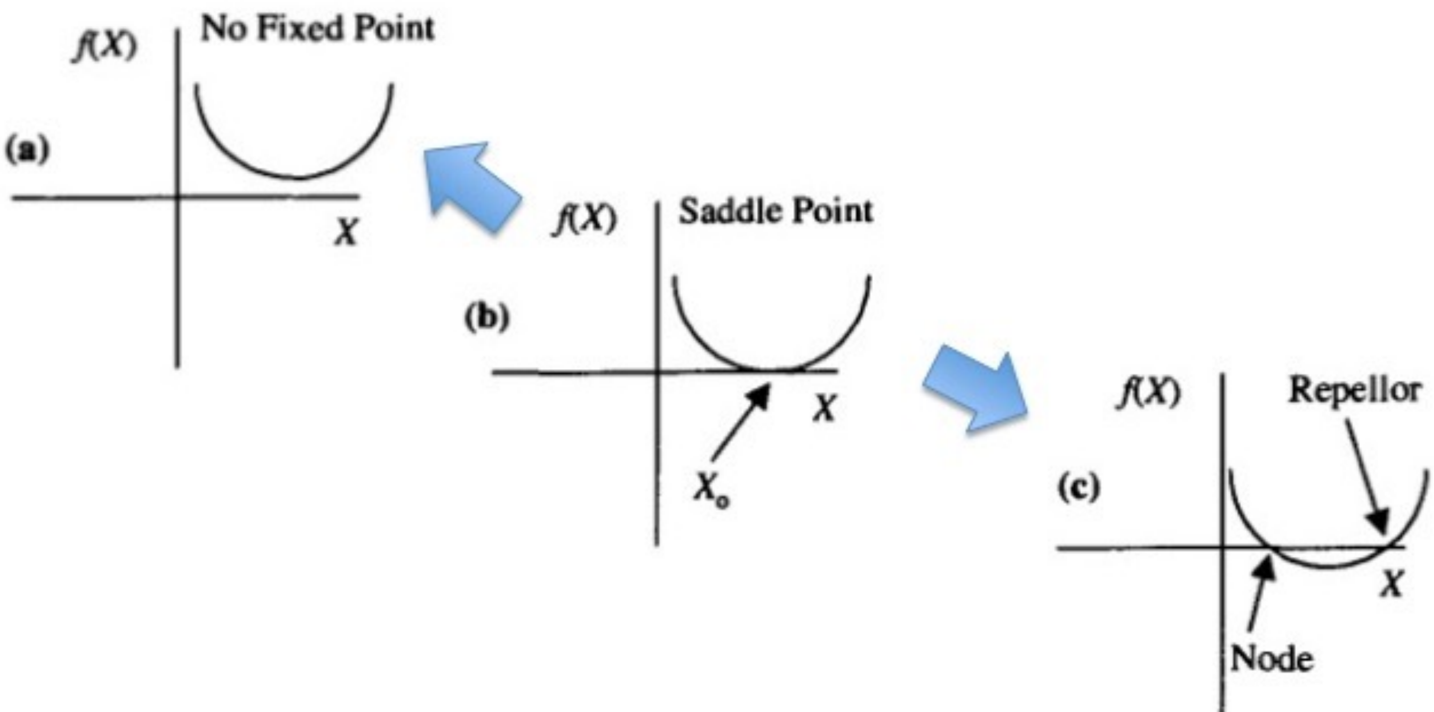
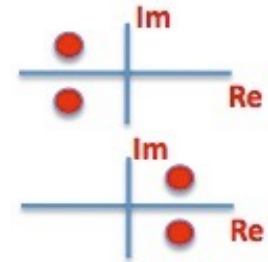


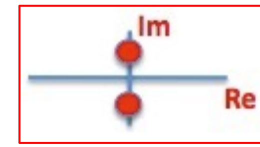
Fig. 3.5. In one-dimensional state spaces, a saddle point, the point X_o in (b), is structurally unstable. A small change in the function $f(X)$, for example pushing it up or down along the vertical axis, either removes the fixed point (a), or changes it into a node and a repellor (c).

Biforcazioni in 2D

Limit Cycle Bifurcations



As we saw earlier, a fixed point in a two-dimensional state space may also have complex-valued characteristic values for which the trajectories have spiral-type behavior. A bifurcation occurs when the characteristic values move from the left-hand side of the complex plane to the right-hand side; that is, the bifurcation occurs when the real part of the characteristic value goes to 0.



We can also have limit cycle behavior in two-dimensional systems. The birth and death of a limit cycle are bifurcation events. The birth of a stable limit cycle is called a ***Hopf bifurcation*** (named after the mathematician E. Hopf). (Although this type of bifurcation was known and understood by Poincaré and later studied by the Russian mathematician A. D. Andronov in the 1930s, Hopf was the first to extend these ideas to higher-dimensional state spaces.) Since we can use a Poincaré section to study a limit cycle and since for a two-dimensional state space, the Poincaré section is just a line segment, the bifurcations of limit cycles can be studied by the same methods used for studying bifurcations of one-dimensional dynamical systems.

Es: BRUSSELLATOR

A Hopf bifurcation can be modeled using the following normal form equations:

Flusso a due dimensioni

$$\dot{x}_1 = -x_2 + x_1\{\mu - (x_1^2 + x_2^2)\} \qquad (3.17\text{-}5a)$$

$$\dot{x}_2 = +x_1 + x_2\{\mu - (x_1^2 + x_2^2)\} \qquad (3.17\text{-}5b)$$

$$\dot{x}_1 = -x_2 + x_1\{\mu - (x_1^2 + x_2^2)\} \qquad (3.17\text{-}5a)$$

$$\dot{x}_2 = +x_1 + x_2\{\mu - (x_1^2 + x_2^2)\} \qquad (3.17\text{-}5b)$$

Esiste chiaramente un punto fisso nell'origine...

The geometric form of the trajectories is clearer if we change from (x_1, x_2) coordinates to polar coordinates (r,θ) defined in the following equations and illustrated in Fig. 3.18.

$$r = \sqrt{(x_1^2 + x_2^2)}$$

Distanza dal punto fisso nell'origine

$$\tan\theta = \frac{x_2}{x_1} \qquad (3.17\text{-}6)$$

Using these polar coordinates, we write Eqs. (3.17-5) as

$$\dot{r} = r\{\mu - r^2\} \equiv f(r) \qquad (3.17\text{-}7a)$$

cubica

$$\dot{\theta} = 1 \rightarrow \theta(t) = \theta_o + t \qquad (3.17\text{-}7b)$$

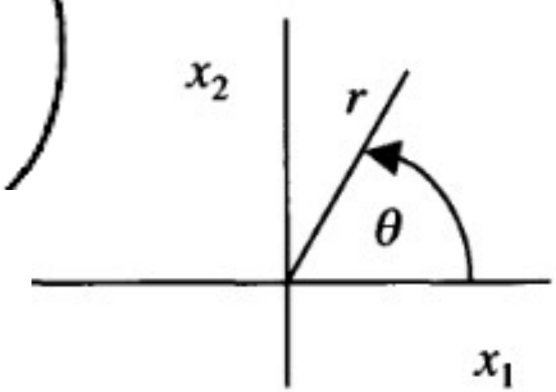


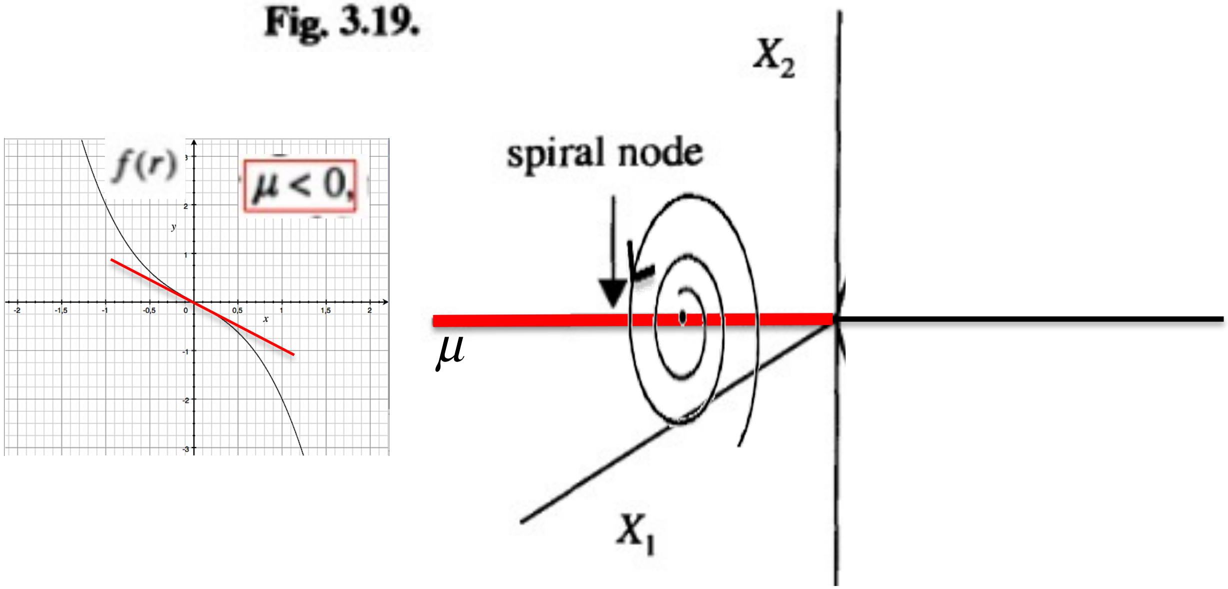
Fig. 3.18. The definition of polar coordinates. r is the length of the radius vector from the origin. θ is the angle between the radius vector and the positive x_1 axis.

Now let us interpret the geometric nature of the trajectories that follow from Eqs. (3.17-7). The solution to Eq. (3.17-7b) is simply

$$\theta(t) = \theta_o + t \tag{3.17-8}$$

that is, the angle continues to increase with time as the trajectory spirals around the origin. For $\mu < 0$, there is just one fixed point for r, namely $r = 0$. By evaluating the derivative of $f(r)$ with respect to r at $r = 0$, we see that the characteristic value is equal to μ. Thus, for $\mu < 0$, that derivative is negative, and the fixed point is stable. In fact, it is a spiral node.

Fig. 3.19.



$$\begin{cases} \dot{r} = r\{\mu - r^2\} \equiv f(r) \\ \dot{\theta} = 1 \end{cases}$$

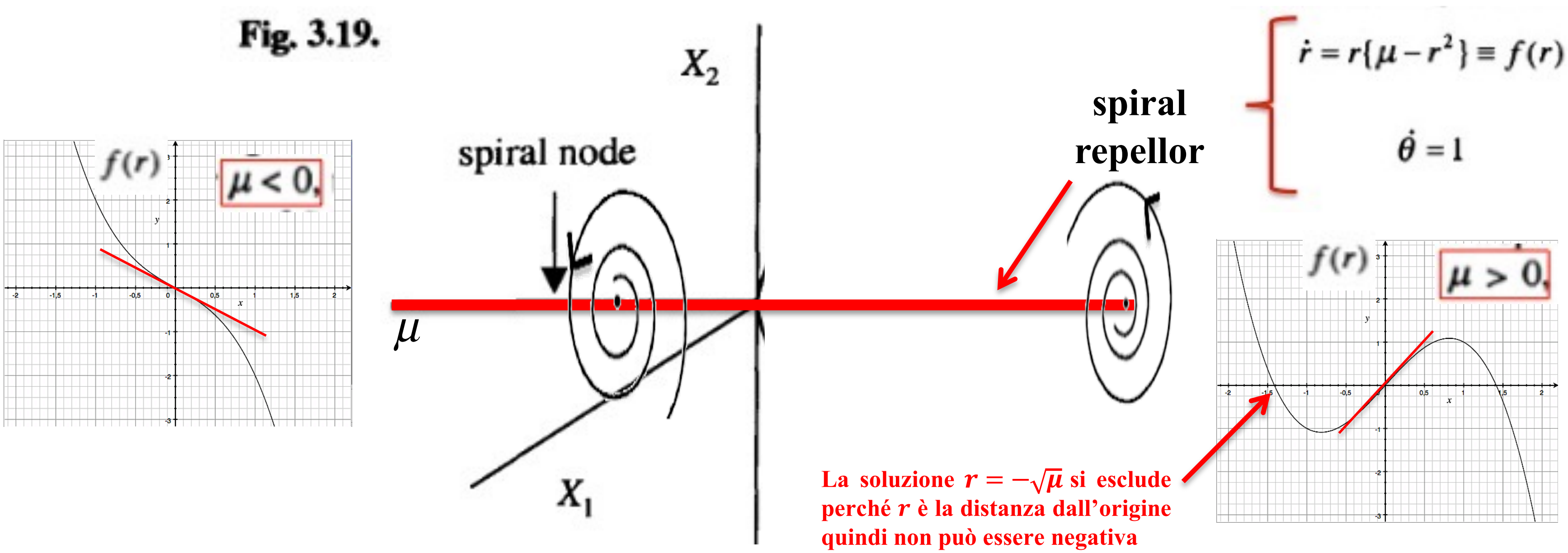
Now let us interpret the geometric nature of the trajectories that follow from Eqs. (3.17-7). The solution to Eq. (3.17-7b) is simply

$$\theta(t) = \theta_o + t \tag{3.17-8}$$

that is, the angle continues to increase with time as the trajectory spirals around the origin. For $\mu < 0$, there is just one fixed point for r, namely $r = 0$. By evaluating the derivative of $f(r)$ with respect to r at $r = 0$, we see that the characteristic value is equal to μ. Thus, for $\mu < 0$, that derivative is negative, and the fixed point is stable. In fact, it is a spiral node.

For $\mu > 0$, the fixed point at the origin is a spiral repellor; it is unstable; trajectories starting near the origin spiral away from it. There is, however, another fixed point for r, namely, $r = \sqrt{\mu}$.

Fig. 3.19.



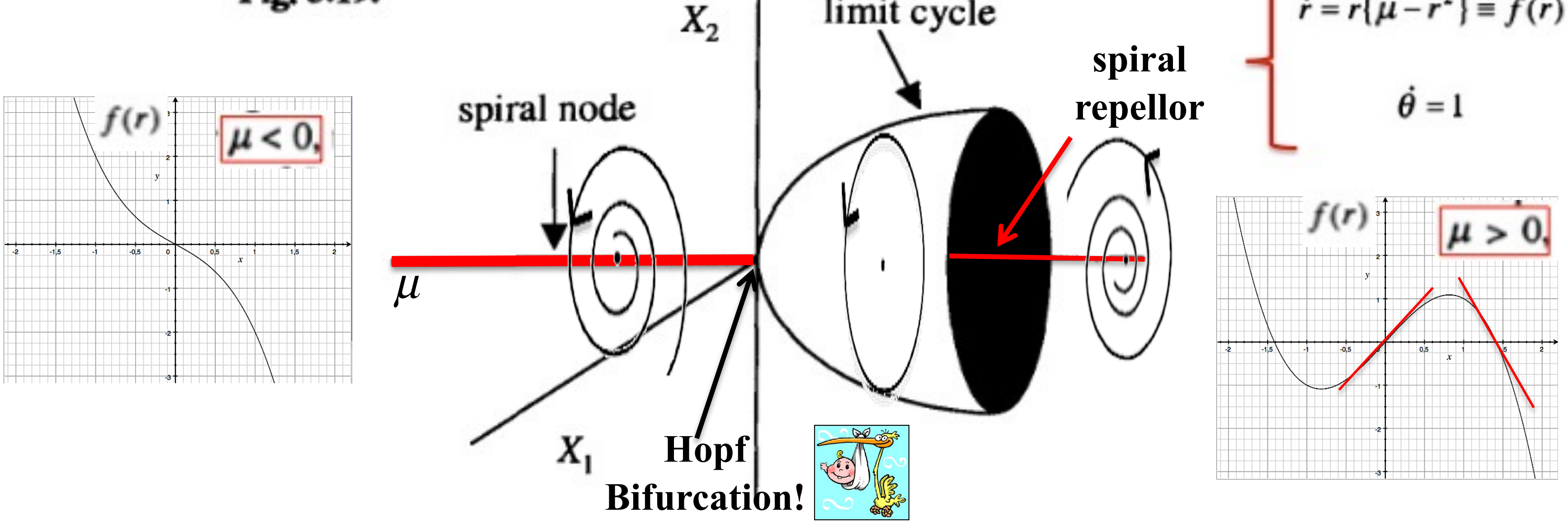
Now let us interpret the geometric nature of the trajectories that follow from Eqs. (3.17-7). The solution to Eq. (3.17-7b) is simply

$$\theta(t) = \theta_o + t \qquad (3.17\text{-}8)$$

that is, the angle continues to increase with time as the trajectory spirals around the origin. For $\mu < 0$, there is just one fixed point for r, namely $r = 0$. By evaluating the derivative of $f(r)$ with respect to r at $r = 0$, we see that the characteristic value is equal to μ. Thus, for $\mu < 0$, that derivative is negative, and the fixed point is stable. In fact, it is a spiral node.

For $\mu > 0$, the fixed point at the origin is a spiral repellor; it is unstable; trajectories starting near the origin spiral away from it. There is, however, another fixed point for r, namely, $r = \sqrt{\mu}$. This fixed point for r corresponds to a limit cycle with a period of 2π [in the time units of Eqs. (3.17-7)]. We say that the limit cycle is born at the bifurcation value $\mu = 0$. Fig. 3.19 shows the bifurcation diagram for the Hopf bifurcation.

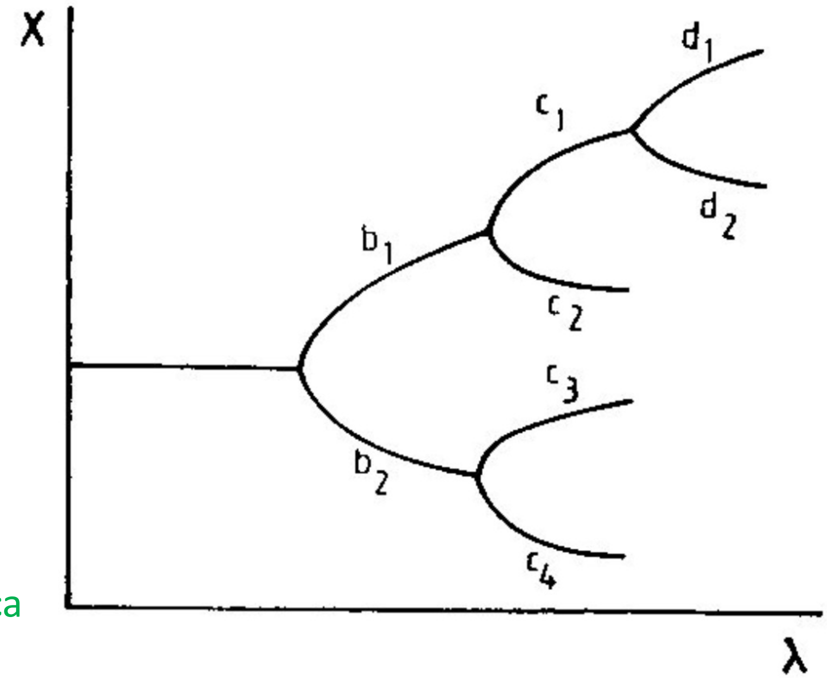
Fig. 3.19.

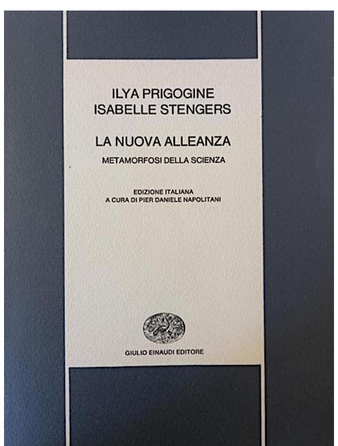


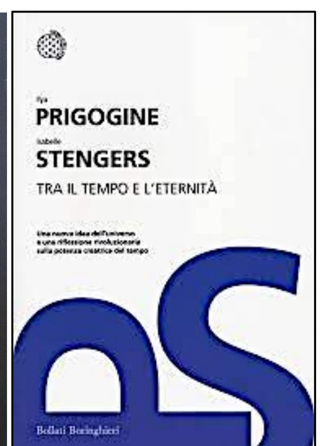
Biforcazioni e Strutture Dissipative

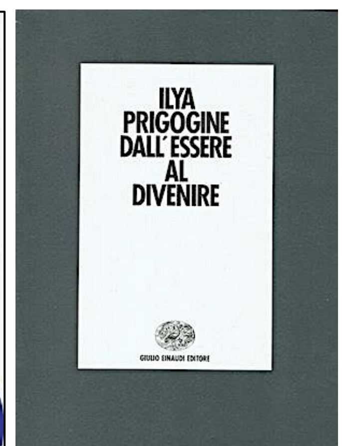
Sequenza di biforcazioni nei sistemi lontani dall'equilibrio

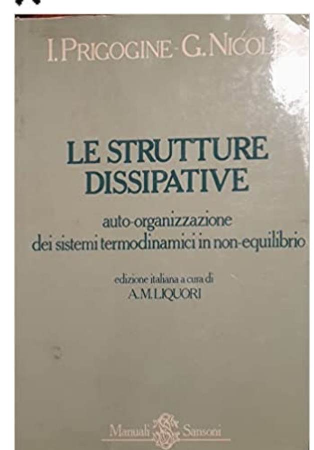
Ilya Prigogine
(1917-2003)
Nobel 1977 per la Chimica

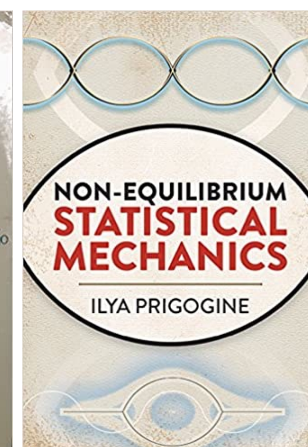












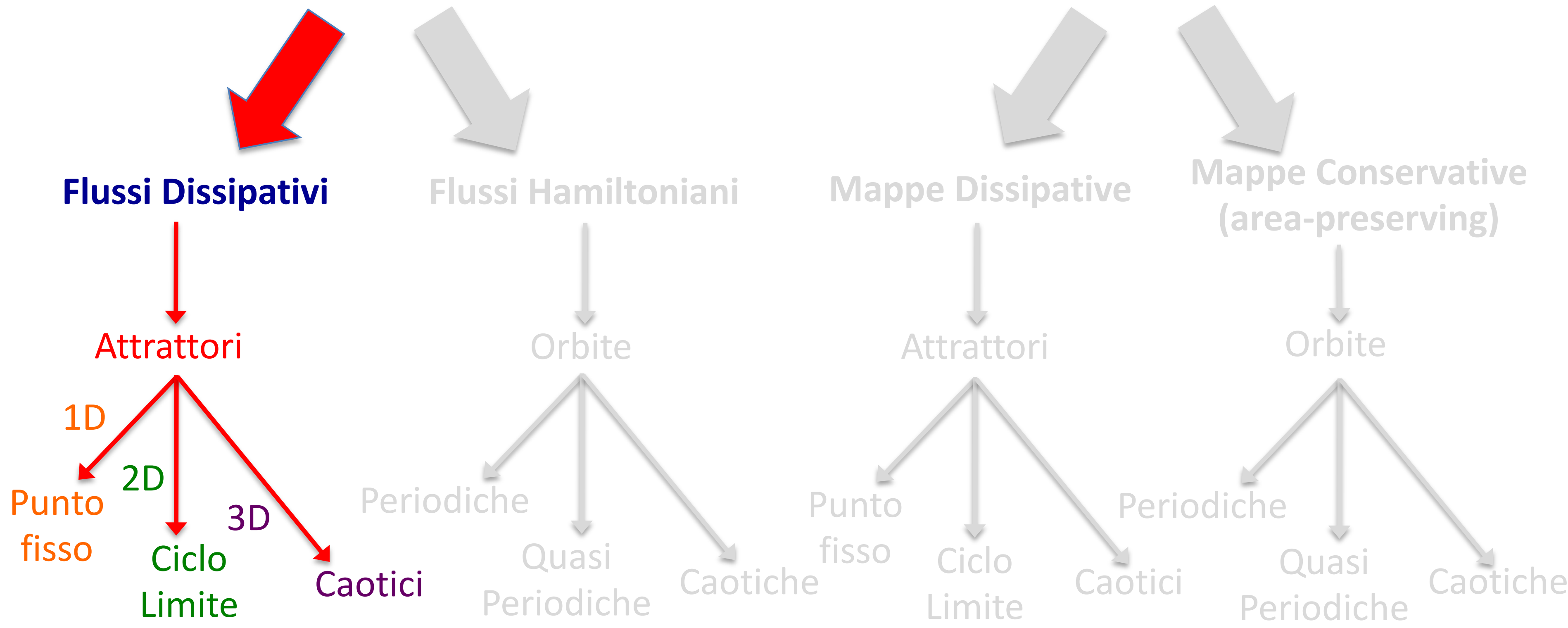
Classificazione dei Sistemi Dinamici

Sistemi dinamici continui (Flussi)

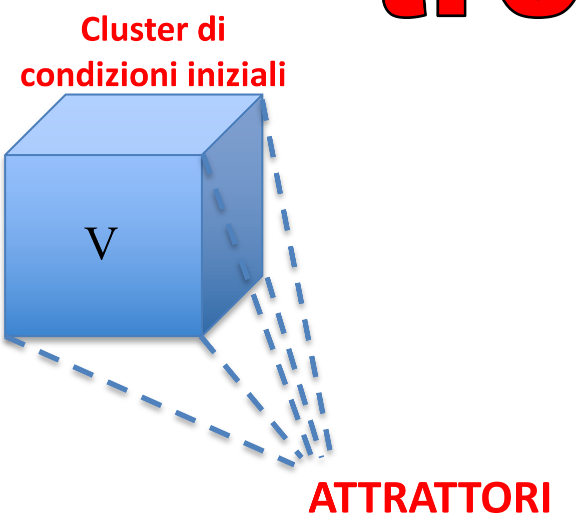
$$\dot{X} = f(X)$$

Sistemi dinamici discreti (Mappe)

$$x_{n+1} = f(x_n)$$



Flussi dissipativi in tre dimensioni

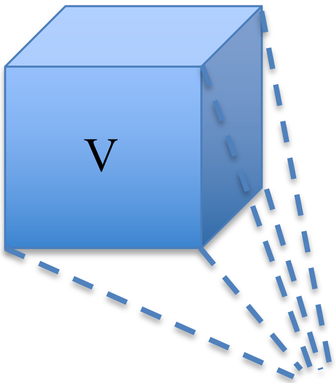


$$\frac{1}{V}\frac{dV}{dt} = \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i} \equiv div(f) < 0$$

- fixed points (dim.0)
- limit cycles (dim.1)
- quasiperiodic attractors (dim.2)
- chaotic attractors (dim…???)

Flussi dissipativi in tre dimensioni

Cluster di condizioni iniziali



$$\frac{1}{V}\frac{dV}{dt} = \sum_{i=1}^{N}\frac{\partial f_i}{\partial x_i} \equiv div(f) < 0$$

ATTRATTORI

- fixed points (dim.0)
- limit cycles (dim.1)
- quasiperiodic attractors (dim.2)
- chaotic attractors (fractal dimension between 2 and 3)

4

Three-Dimensional State Space and Chaos

4.1 Overview

In the previous chapter, we introduced some of the standard methods for analyzing dynamical systems described by systems of ordinary differential equations, but we limited the discussion to state spaces with one or two dimensions. We are now ready to take the important step to three dimensions. This is a crucial step, not because we live in a three-dimensional world (remember that we are talking about state space, not physical space), but because in three dimensions dynamical systems can behave in ways that are not possible in one or two dimensions. Foremost among these new possibilities is chaos.

First we will give a hand-waving argument (we could call it heuristic if we wanted to sound more sophisticated) that shows why chaotic behavior may occur in three dimensions. We will then discuss, in parallel with the treatment of the previous chapter, a classification of the types of fixed points that occur in three dimensions. However, we gradually wean ourselves from the standard analytic techniques and begin to rely more and more on graphic and geometrical (topological) arguments. This change reflects the flavor of current developments in dynamical systems theory. In fact, the main goal of this chapter is to develop geometrical pictures of trajectories, attractors, and bifurcations in three-dimensional state spaces.

4.2 Heuristics

We will describe, in a rather loose way, why three (or more) state space dimensions are needed to have chaotic behavior. First, we should remind ourselves that we are dealing with dissipative systems whose trajectories eventually approach an attractor. For the moment we are concerned only with the trajectories that have settled into the attracting region of state space. When we write about the divergence of nearby trajectories, we are concerned with the behavior of trajectories within the attracting region of state space.

In a somewhat different context we will need to consider sensitive dependence on initial conditions. Initial conditions that are not, in general, part of an attractor can lead to very different long-term behaviors on different attractors. Those behaviors, determined by the nature of the attractor (or attractors), might be time-independent or periodic or chaotic.

As we saw in Chapter 1, chaotic behavior is characterized by the divergence of nearby trajectories in state space. As a function of time, the "separation" (suitably defined) between two nearby trajectories increases exponentially, at least for short times. The last restriction is necessary because we are concerned with systems whose trajectories stay within some bounded region of state space. The system does not "blow up." There are three requirements for chaotic behavior in such a situation:

1. no intersection of different trajectories;
2. bounded trajectories;
3. exponential divergence of nearby trajectories.

These conditions cannot be satisfied simultaneously in one- or two-dimensional state spaces. You should convince yourself that this is true by sketching some trajectories in a two-dimensional state space on a sheet of paper. However, in three dimensions, initially nearby trajectories can continue to diverge by wrapping over and under each other. Obviously sketching three-dimensional trajectories is more difficult. You might try using some relatively stiff wire to form some trajectories in three dimensions to show that all three requirements for chaotic behavior can be met. You should quickly discover that these requirements lead to trajectories that initially diverge, then curve back through the state space, forming in the process an intricate layered structure. Figure 4.1 is a sketch of diverging trajectories in a three-dimensional state space.

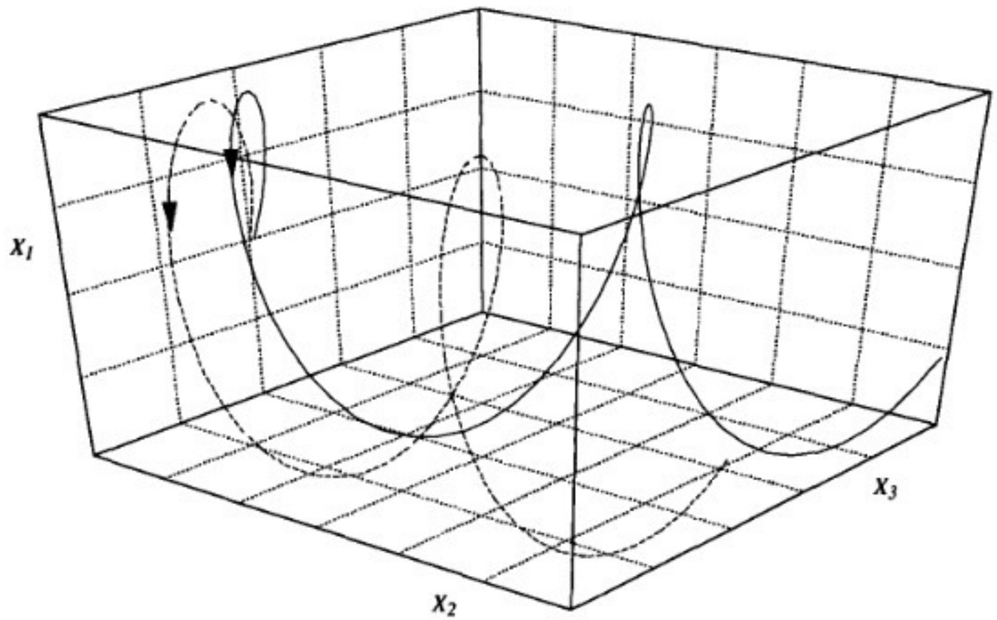


Fig. 4.1. A sketch of trajectories in a three–dimensional state space. Notice how two nearby trajectories can continue to behave quite differently from each other yet remain bounded by weaving in and out and over and under each other.

The notion of exponential divergence of nearby trajectories is made formal by introducing the ***Lyapunov exponent***. If two nearby trajectories on a chaotic attractor start off with a separation d_0 at time $t = 0$, then the trajectories diverge so that their separation at time t, denoted by $d(t)$, satisfies the expression

$$d(t) = d_0 e^{\lambda t} \tag{4.2-1}$$

The parameter λ in Eq. (4.2-1) is called the Lyapunov exponent for the trajectories. If λ is positive, then we say the behavior is chaotic. (Section 4.13 takes up the question of Lyapunov exponents in more detail.) From this definition of chaotic behavior, we see that chaos is a property of a collection of trajectories.

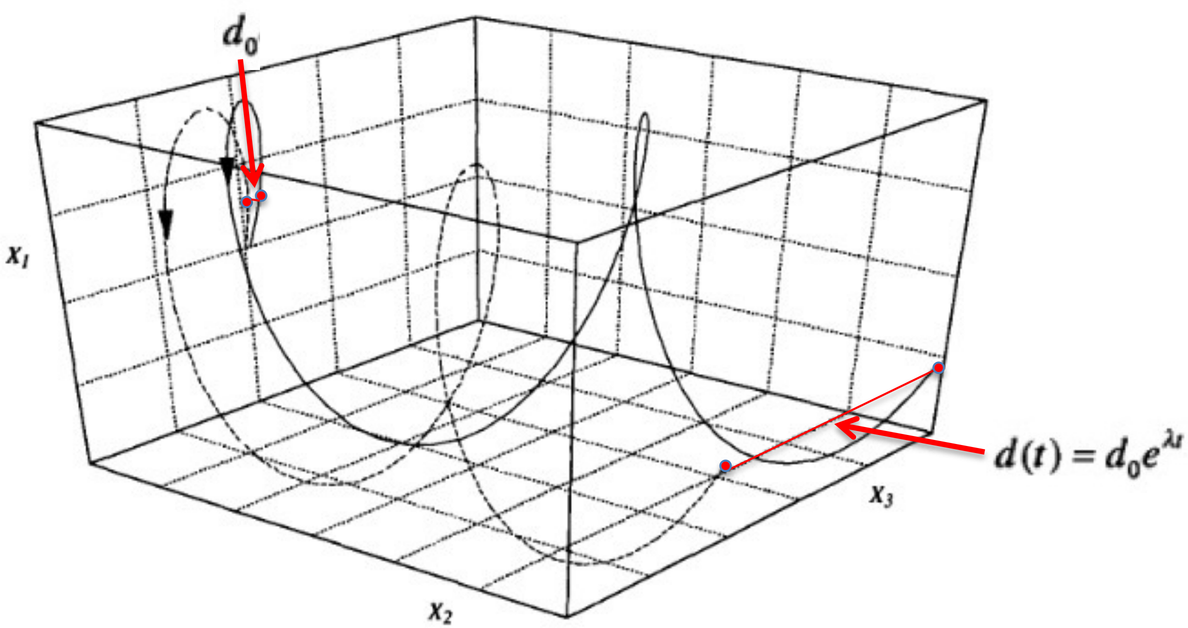
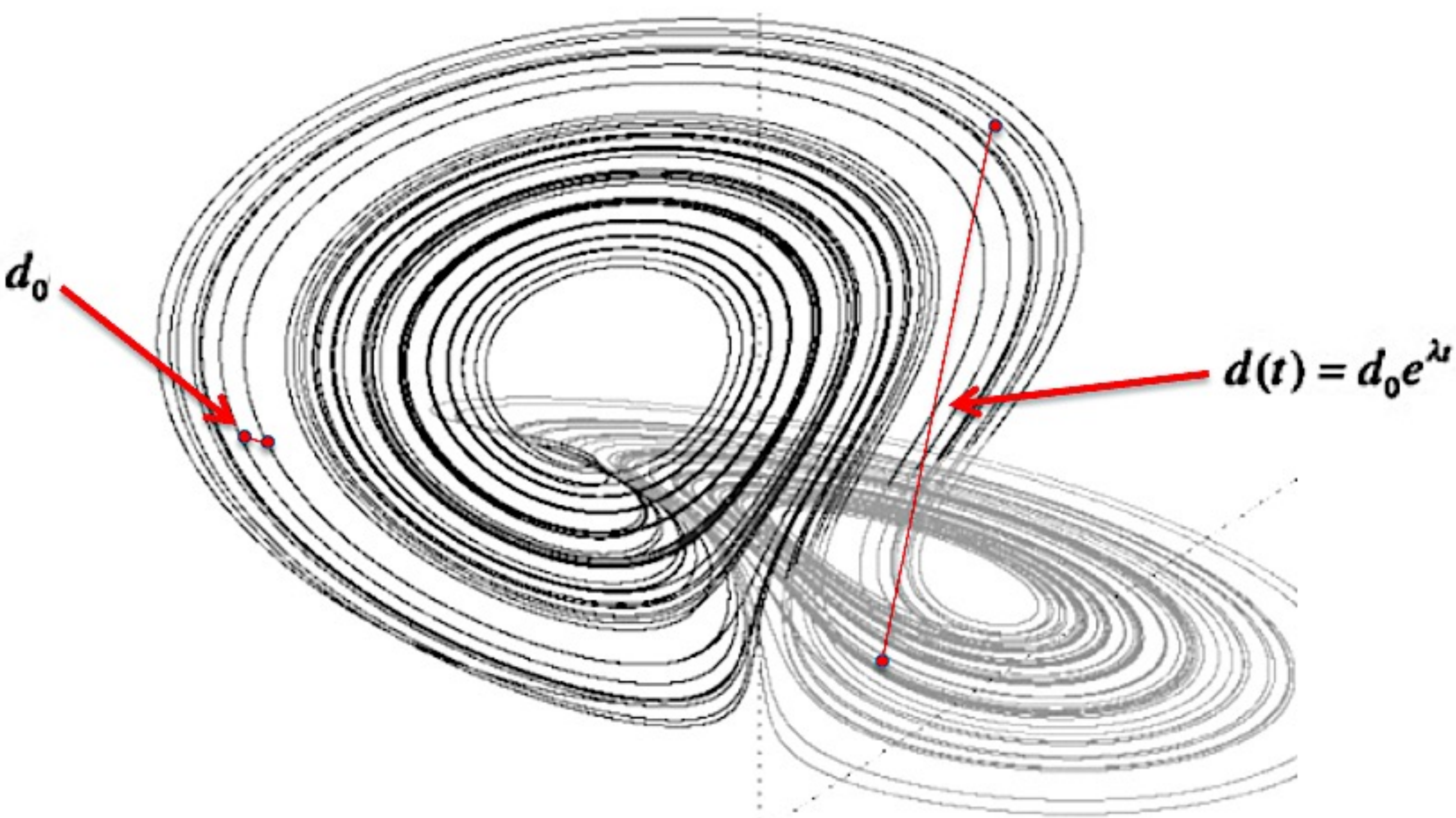
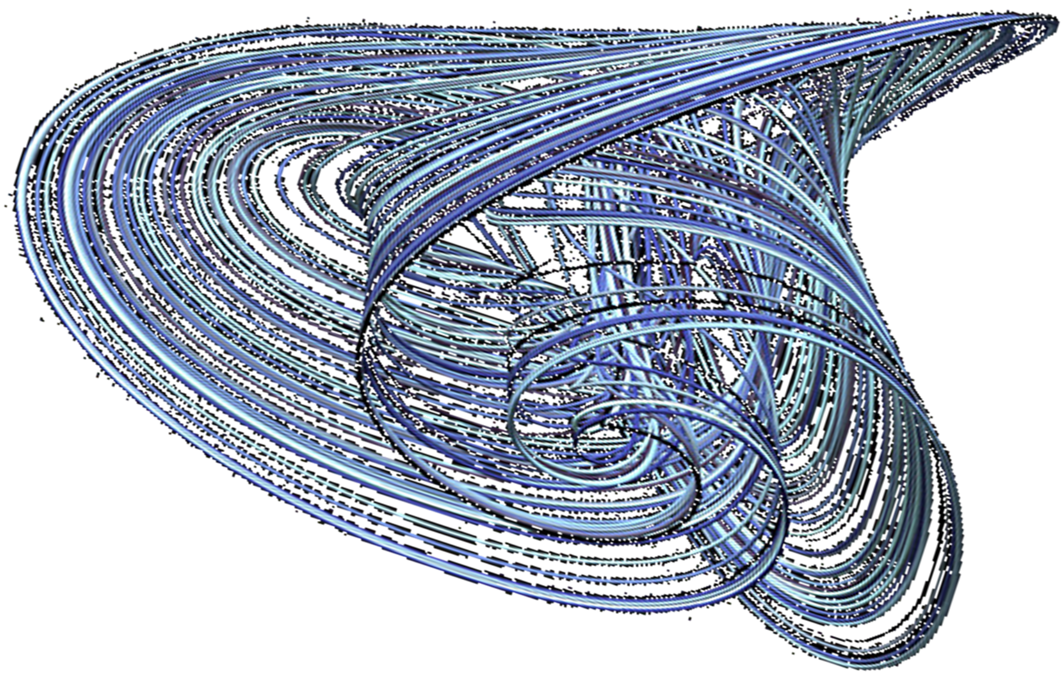


Fig. 4.1. A sketch of trajectories in a three-dimensional state space. Notice how two nearby trajectories can continue to behave quite differently from each other yet remain bounded by weaving in and out and over and under each other.

Chaos, however, also appears in the behavior of a single trajectory. As the trajectory wanders through the (chaotic) attractor in state space, it will eventually return near some point it previously visited. (Of course, it cannot return exactly to that point. If it did, then the trajectory would be periodic.) If the trajectories exhibit exponential divergence, then the trajectory on its second visit to a particular neighborhood will have subsequent behavior, quite different from its behavior on the first visit. Thus, the impression of the time record of this behavior will be one of nonreproducibility, nonperiodicity, in short, of chaos.



The crucial feature of state space with three or more dimensions that permits chaotic behavior is the ability of trajectories to remain within some bounded region by intertwining and wrapping around each other (without intersecting!) and without repeating themselves exactly. Clearly the geometry associated with such trajectories is going to be strange. In fact, such attractors are now called ***strange attractors***. In Chapter 9, we will give a more precise definition of a strange attractor in terms of the notion of fractal dimension. If the behavior on the attractor is chaotic, that is, if the trajectories on the attractor display exponential divergence of nearby trajectories (on the average), then we say the attractor is chaotic. Many authors use the terms *strange attractor* and ***chaotic attractor*** interchangeably, but in principle they are distinct [GOP84].



4.4 Three-Dimensional Dynamical Systems

We will now introduce some of the formalism for the description of a dynamical system with three state variables. We call a dynamical system three-dimensional if it has three independent dynamical variables, the values of which at a given instant of time uniquely specify the state of the system. We assume that we can write the time-evolution equations for the system in the form of the standard set of first-order ordinary differential equations. (Dynamical systems modeled by iterated map functions will be discussed in Chapter 5.) Here we will use x with a subscript 1, 2, or 3 to identify the variables. This formalism can then easily be generalized to any number of dimensions simply by increasing the numerical range of the subscripts. The differential equations take the form

$$\dot{X} = p(Y - X)$$
$$\dot{Y} = -XZ + rX - Y$$
$$\dot{Z} = XY - bZ$$

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, x_3) \\ \dot{x}_2 = f_2(x_1, x_2, x_3) \\ \dot{x}_3 = f_3(x_1, x_2, x_3) \end{cases} \qquad (4.4\text{-}1)$$

The Lorenz model equations of Chapter 1 are of this form. Note that the three functions f_1, f_2, and f_3 do not involve time explicitly; again, we say that the system is autonomous.

As an aside, we note that some authors like to use a symbolic "vector" form to write the system of equations:

$$\dot{\vec{x}} = \vec{f}(\vec{x}) \qquad (4.4\text{-}2)$$

Here $\vec{x}$ stands for the three symbols $x_1, x_2 x_3$, and $\vec{f}$ stands for the three functions on the right-hand side of Eqs. (4.4-1).

The differential equations describing two-dimensional systems subject to a time-dependent "force" (and hence nonautonomous) can also be written in the form of Eq. (4.4-1) by making use of the "trick" introduced in Chapter 3: Suppose that the two-dimensional system is described by equations of the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, t) \\ \dot{x}_2 &= f_2(x_1, x_2, t) \end{aligned} \tag{4.4-3}$$

The trick is to introduce a third variable, $x_3 = t$. The three "autonomous" equations then become

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, x_3) \\ \dot{x}_2 &= f_2(x_1, x_2, x_3) \\ \dot{x}_3 &= 1 \end{aligned} \tag{4.4-4}$$

which are of the same form as Eq. (4.4-1). As we shall see, this trick is particularly useful when the time-dependent term is periodic in time.

Exercise 4.4-1. The "forced" van der Pol equation is used to describe an electronic triode tube circuit subject to a periodic electrical signal. The equation for $q(t)$, the charge oscillating in the circuit, can be put in the form

$$\frac{d^2q}{dt^2} + \gamma(q)\frac{dq}{dt} + q(t) = g \sin \omega t$$

Use the trick introduced earlier to write this equation in the standard form of Eq. (4.4-1).

4.5 Fixed Points in Three Dimensions (dim = 0)

The fixed points of the system of Eqs. (4.4-1) are found, of course, by setting the three time derivatives equal to 0. [Two-dimensional forced systems, even if written in the three-dimensional form (4.4-4), do not have any fixed points because, as the last of Eqs. (4.4-4) shows, we never have $\dot{x}_3 = t = 0$. Thus, we will need other techniques to deal with them.] The nature of each of the fixed points is determined by the three characteristic values of the Jacobian matrix of partial derivatives evaluated at the fixed point in question. The Jacobian matrix is

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{pmatrix} \tag{4.5-1}$$

In finding the characteristic values of this matrix, we will generally have a cubic equation, whose roots will be the three characteristic values labeled $\lambda_1, \lambda_2, \lambda_3$.

Some mathematical details: The standard theory of cubic equations tells us that a cubic equation of the form

$$\lambda^3 + p\lambda^2 + q\lambda + r = 0 \tag{4.5-2}$$

can be changed to the "standard" form

$$x^3 + ax + b = 0 \tag{4.5-3}$$

by the use of the substitutions

$$\begin{aligned} x &= \lambda + p/3 \\ a &= \frac{1}{3}(3q - p^2) \\ b &= \frac{1}{27}(2p^3 - 9qp + 27r) \end{aligned} \tag{4.5-4}$$

If we now introduce

$$\begin{aligned} s &= \left(\frac{b^2}{4} + \frac{a^3}{27}\right) \\ A &= (-b/2 + \sqrt{s})^{\frac{1}{3}} \\ B &= (-b/2 - \sqrt{s})^{\frac{1}{3}} \end{aligned} \tag{4.5-5}$$

the three roots of the x equation can be written as

$$\begin{aligned}
\lambda_1 &= A+B \\
\lambda_2 &= -\left(\frac{A+B}{2}\right)+\left(\frac{A-B}{2}\right)\sqrt{-3} \\
\lambda_3 &= -\left(\frac{A+B}{2}\right)-\left(\frac{A-B}{2}\right)\sqrt{-3}
\end{aligned} \qquad (4.5\text{-}6)$$

from which the characteristic values for the matrix can be found by working back through the set of substitutions. Most readers will be greatly relieved to know that we will not make explicit use of these equations. But it is important to know the form of the solutions.

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\lambda_1 &= A+B \\
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\lambda_3 &= -\left(\frac{A+B}{2}\right)-\left(\frac{A-B}{2}\right)\sqrt{-3}
\end{aligned} \tag{4.5-6}$$

from which the characteristic values for the matrix can be found by working back through the set of substitutions. Most readers will be greatly relieved to know that we will not make explicit use of these equations. But it is important to know the form of the solutions.

There are three cases to consider:

"standard" form

$x^3 + ax + b = 0$

$s = \left(\frac{b^2}{4} + \frac{a^3}{27}\right)$

1. The three characteristic values are real and unequal ($s < 0$).
2. The three characteristic values are real and at least two are equal ($s = 0$).
3. There is one real characteristic value and two complex conjugate values ($s > 0$).

Case 2 is just a borderline case and need not be treated separately.

Punti Fissi in uno Spazio degli Stati a Tre Dimensioni

The four basic types of fixed points for a three-dimensional state space are:

1. ***Node***. All the characteristic values are real and negative. All trajectories in the neighborhood of the node are attracted toward the fixed point without looping around the fixed point.

   1s. ***Spiral Node***. All the characteristic values have negative real parts but two of them have nonzero imaginary parts (and in fact form a complex conjugate pair). The trajectories spiral around the node on a "surface" as they approach the node.

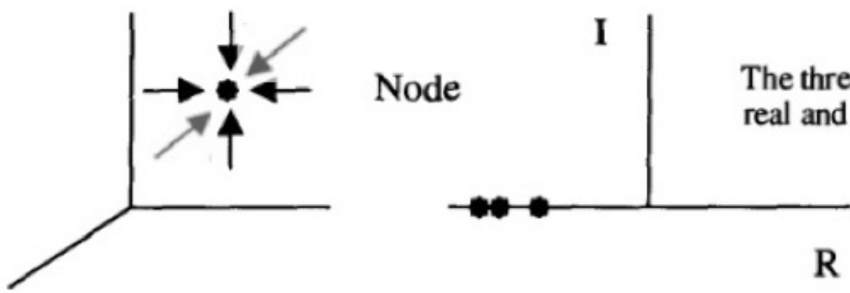
Equazione caratteristica:

$$\lambda^3 + p\lambda^2 + q\lambda + r = 0$$

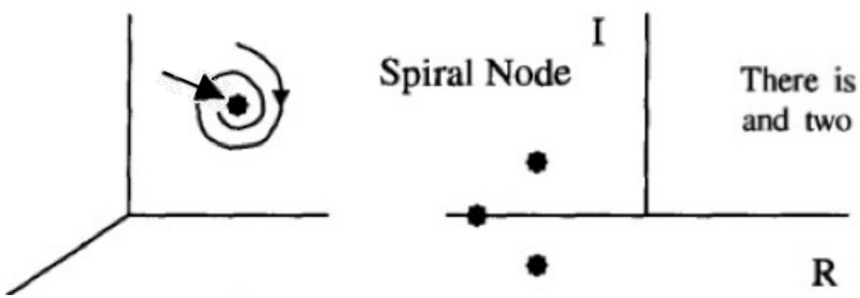
"standard" form

$$x^3 + ax + b = 0$$

$$s = \left(\frac{b^2}{4} + \frac{a^3}{27}\right)$$



The three characteristic values are real and unequal ($s < 0$).



There is one real characteristic value and two complex conjugate values ($s > 0$).

Punti Fissi in uno Spazio degli Stati a Tre Dimensioni

The four basic types of fixed points for a three-dimensional state space are:

2. ***Repellor***. All the characteristic values are real and positive. All trajectories in the neighborhood of the repellor diverge from the repellor.

2s. ***Spiral Repellor***. All the characteristic values have positive real parts, but two of them have nonzero imaginary parts (and in fact form a complex conjugate pair). Trajectories spiral around the repellor (on a "surface") as they are repelled from the fixed point.

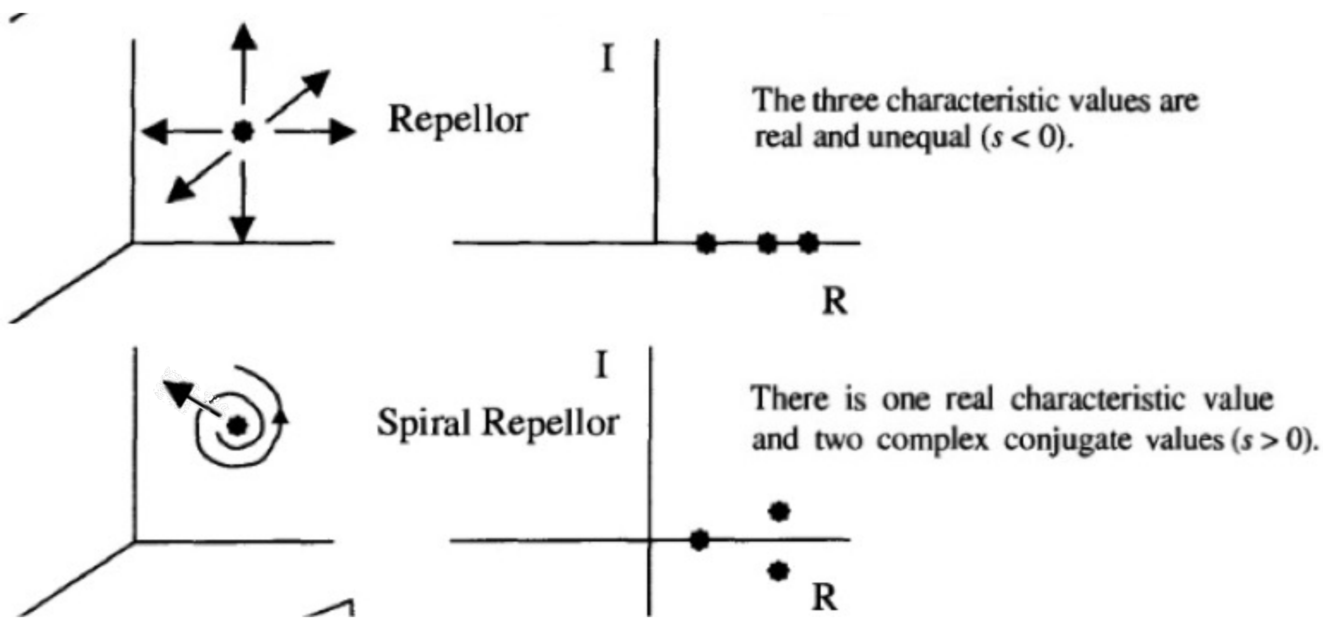
Equazione caratteristica:

$$\lambda^3 + p\lambda^2 + q\lambda + r = 0$$

"standard" form

$$x^3 + ax + b = 0$$

$$s = \left(\frac{b^2}{4} + \frac{a^3}{27}\right)$$



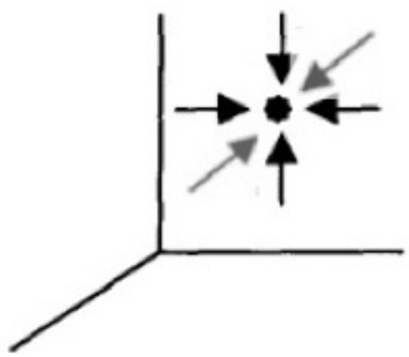
Punti Fissi in uno Spazio degli Stati a Tre Dimensioni

For state spaces with three or more dimensions, it is common to specify the so-called *index* of a fixed point.

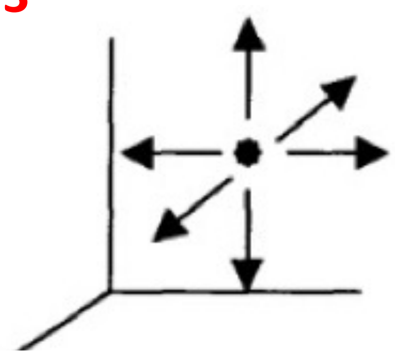
> The ***index*** of a fixed point is defined to be the number of characteristic values of that fixed point whose real parts are positive.

In more geometric terms, the index is equal to the spatial dimension of the out-set of that fixed point. For a node (which does not have an out-set), the index is equal to 0. For a repellor, the index is equal to 3 for a three-dimensional state space. A saddle point can have either an index of 1, if the out-set is a curve, or an index of 2, if the out-set is a surface as shown in Fig. 4.3.

Index = 0



Index = 3



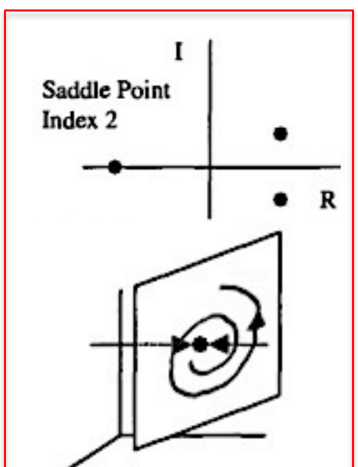
Punti Fissi in uno Spazio degli Stati a Tre Dimensioni

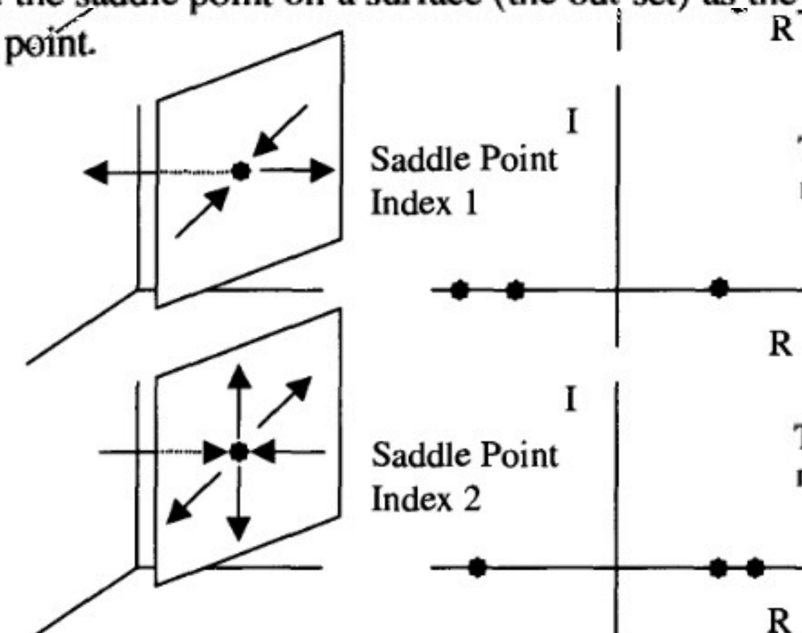
3. ***Saddle point — index–1.*** All characteristic values are real. One is positive and two are negative. Trajectories approach the saddle point on a surface (the in-set) and diverge along a curve (the out-set).

   3s. ***Spiral Saddle Point — index–1.*** The two characteristic values with negative real parts form a complex conjugate pair. Trajectories spiral around the saddle point as they approach on the in-set surface.

4. ***Saddle point — index–2.*** All characteristic values are real. Two are positive and one is negative. Trajectories approach the saddle point on a curve (the in-set) and diverge from the saddle point on a surface (the out-set).

   4s. ***Spiral Saddle Point — index–2.*** The two characteristic values with positive real parts form a complex conjugate pair. Trajectories spiral around the saddle point on a surface (the out-set) as they diverge from the saddle point.





The three characteristic values are real and unequal ($s < 0$).

The three characteristic values are real and unequal ($s < 0$).

4.6 Limit Cycles and Poincaré Sections (dim = 1)

As we saw in Chapter 3, dynamical systems in two (and higher) dimensions can also settle into long-term behavior associated with repetitive, periodic limit cycles. We also learned that the Poincaré section technique can be used to reduce the dimensionality of the description of these limit cycles and to make their analysis simpler.

First, we focus on the construction of a Poincaré section for the system. For a three-dimensional state space, the Poincaré section is generated by choosing a ***Poincaré plane*** (a two-dimensional surface) and recording on that surface the points at which a given trajectory cuts through that surface. (In most cases the choice of plane is not crucial as long as the trajectories cut the surface ***transversely***, that is, the trajectories do not run parallel or almost parallel to the surface as they pass through; see Fig. 4.4.) For autonomous systems, such as the Lorenz model equations, we choose some convenient plane in the state space, say, the *XY* plane for the Lorenz equations. When a trajectory crosses that plane passing from, for example, negative *Z* values to positive *Z* values, we record that crossing point.

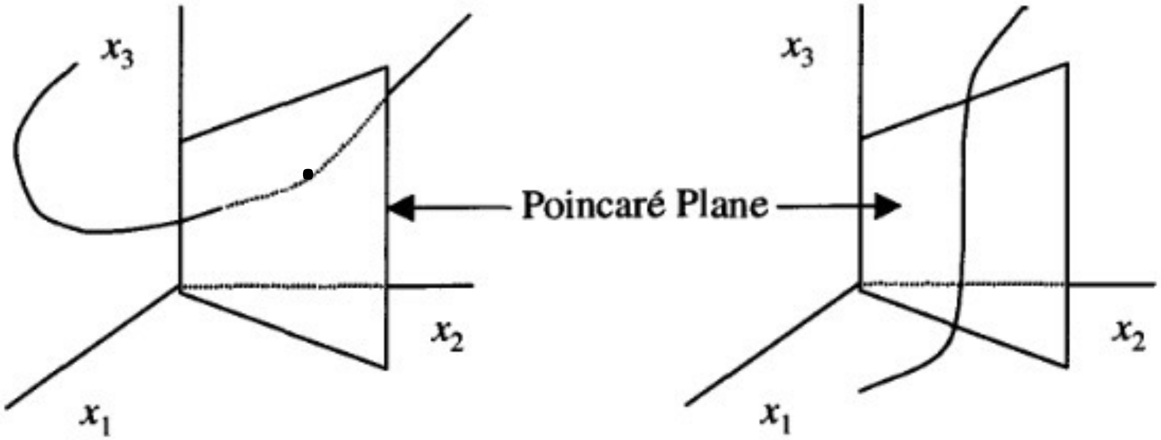


Fig. 4.4. A Poincaré section for a three-dimensional state space. On the left the trajectory crosses the Poincaré plane transversely. On the right the intersection is not transverse because the trajectory runs parallel to the plane for some distance.

In later discussions, it will be useful to indicate on the Poincaré section the record of trajectory intersections with the plane as trajectories approach or diverge from the periodic points. For example, Fig. 4.6 shows a sequence of points $P_0, P_1, P_2, \ldots$ as a trajectory approaches an attracting limit cycle in a three-dimensional state space. (Compare Fig. 4.6 with Fig. 3.13.) The reader should be warned that in some diagrams found in the literature this series of dots will be connected with a smooth curve intersecting (x_1^*, x_2^*). It is important to remember that this curve is not a trajectory. In fact the Poincaré intersection of any single trajectory is just a sequence of points as shown in Fig. 4.6. If a smooth curve is drawn on this kind of diagram, it represents the intersection points of an infinite family of trajectories, all of which are approaching (x_1^*, x_2^*). Later we shall see cases in which such curves intersect. It is important to remember that this intersection does not violate the No-Intersection Theorem because the intersecting curves in this case are not themselves trajectories.

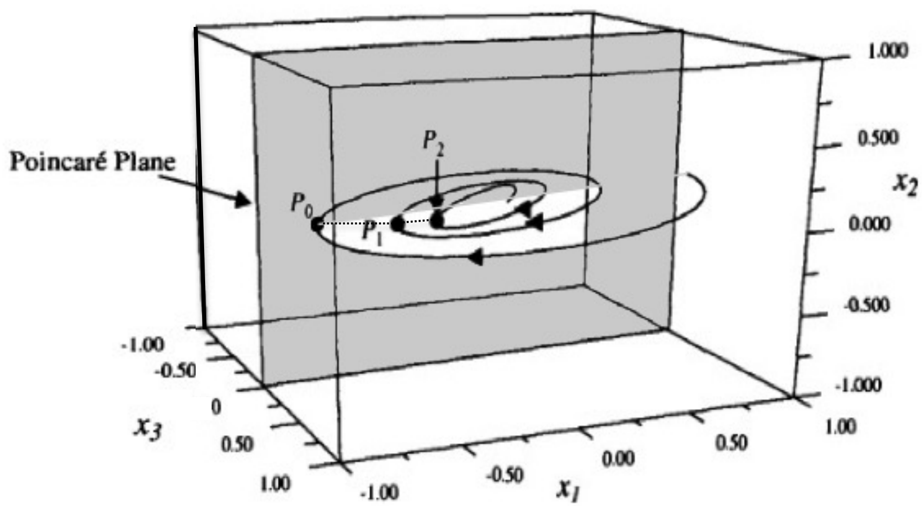


Fig. 4.6. The sequence of points $P_0, P_1, P_2, \ldots$ is the record of successive intersections of a single trajectory with the Poincaré plane (the plane with $x_3 = 0$) as the trajectory goes from $x_3 > 0$ to $x_3 < 0$.

PER CHI VOLESSE APPROFONDIRE IL RUOLO DI POINCARE' COME PRECURSORE DELLA TEORIA DEL CAOS...

(Tre) corpi al margine del caos

Di: **Alessandro Pluchino**

4 Gennaio 2022

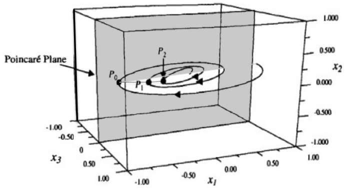
https://www.vitapensata.eu/2022/01/04/tre-corpi-al-margine-del-caos/

Come è noto, il Tre è spesso considerato il *numero perfetto* da diversi punti di vista: dal punto di vista *matematico* costituisce la sintesi del pari (due) e del dispari (uno); dal punto di vista *esoterico* è il simbolo della Grande Triade (Cielo, Terra, Uomo); infine, dal punto di vista *religioso*, rappresenta la perfezione divina (si pensi alla Trinità del Cristianesimo o alla Trimurti induista). Pochi forse sanno, però, che allo stesso tempo il tre rappresenta anche la *soglia dell'imperfezione*, il numero magico che ha condotto la fisica moderna al confine tra ordine e disordine, in quella strana regione oggi conosciuta come "Margine del Caos", spalancando così le porte alla nuova Scienza della Complessità. E la scintilla da cui questa rivoluzione concettuale è partita riguardava un problema di corpi. Per la precisione, appunto, di tre corpi.

Tutto cominciò la notte tra il 31 agosto e il primo settembre del 1879 in una miniera di carbone di Magny, nella Borgogna francese. Alle 3.45 circa del mattino un'esplosione improvvisa scosse la miniera, ustionando e uccidendo gran parte della squadra di ventidue minatori che si trovavano al lavoro a quell'ora. Fu soltanto la perizia e l'acume scientifico di un giovane ingegnere incaricato delle indagini a permettere di risalire alla causa prima dell'esplosione: si era trattato di una lampada perforata accidentalmente che aveva lasciato uscire la fiamma da cui poi, a contatto con un'atmosfera ricca di metano come quella della miniera, aveva avuto inizio il processo che avrebbe portato alla conflagrazione. Quel giovane ingegnere, appena venticinquenne, si chiamava Jules-Henri Poincaré, colui che più avanti si sarebbe distinto come uno dei più grandi matematici e fisici di fine Ottocento (all'epoca si poteva essere ingegnere, matematico e fisico allo stesso tempo!) e che è considerato oggi uno dei padri della teoria dei sistemi dinamici e il precursore assoluto della moderna teoria del Caos. Sarà lui il principale protagonista della storia che stiamo per raccontarvi.

MAPPA DI POINCARE' 2D PER LO STUDIO DELLA STABILITA' DEI CICLI LIMITE IN 3D

We now return to the general discussion of limit cycles. The stability of the limit cycle is determined by a generalization of the Poincaré multipliers introduced in the previous chapter. We assume that the uniqueness of the solutions to the equations used to describe the dynamical system entails the existence of a Poincaré map function (or in the present case, a pair of Poincaré map functions), which relate the coordinates of one point at which the trajectory crosses the Poincaré plane to the coordinates of the next (in time) crossing point. (Again we assume we have chosen a definite crossing sense; e.g., from top to bottom, or from left to right.) These functions take the form



$$\begin{aligned} x_1^{(n+1)} &= F_1(x_1^{(n)}, x_2^{(n)}) \\ x_2^{(n+1)} &= F_2(x_1^{(n)}, x_2^{(n)}) \end{aligned} \qquad (4.6\text{-}1)$$

mappa di Poincarè 2 dim

where the parenthetical superscript indicates the crossing point number.

Here these Poincaré map functions have arisen from the consideration of a Poincaré section for trajectories arising from a set of differential equations. In Chapter 5, we shall consider such map functions as interesting models in their own right, independent of this particular heritage.

The fixed points of the Poincaré section are those points that satisfy

$$\begin{aligned} x_1^* &= F_1(x_1^*, x_2^*) \\ x_2^* &= F_2(x_1^*, x_2^*) \end{aligned} \qquad (4.6\text{-}2)$$

Each fixed point in the Poincaré section corresponds to a limit cycle in the full three-dimensional state space.

MAPPA DI POINCARE' 2D PER LO STUDIO DELLA STABILITA' DEI CICLI LIMITE IN 3D

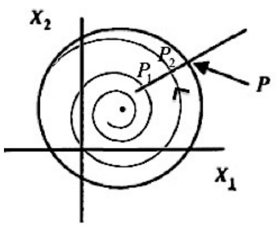
We can characterize the stability of these fixed points by finding the characteristic values of the associated Jacobian matrix of derivatives [sometimes called the ***Floquet matrix***, after Gaston Floquet (1847–1920), a French mathematician who studied, among other things, the properties of differential equations with periodic terms]. This matrix is analogous to the Jacobian matrix used to determine the characteristic values of a fixed point in the full state space. The Jacobian matrix JM is given by

mappa 1 dim

$$d_2 = Md_1$$

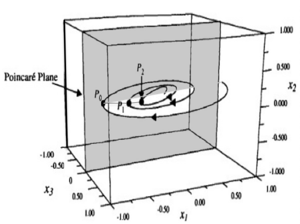
$$d_{n+1} = M^n d_1$$

$$M = \left.\frac{dF}{dP}\right|_{P^*}$$



mappa 2 dim

$$JM = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{pmatrix} \longrightarrow \begin{matrix} M_1 \\ M_2 \end{matrix} \qquad (4.6\text{-}3)$$



where the matrix is to be evaluated at the Poincaré map fixed point in question. The characteristic values of this matrix determine the stability of the limit cycle. A stable limit cycle attracts nearby trajectories, while an unstable limit cycle repels nearby trajectories. In principle, we can use the mathematical methods given in Chapter 3 to find these characteristic values. In practice, however, we most often cannot find these characteristic values explicitly, since, to do that, we would need to know the exact form of the Poincaré map function, and in most cases, we do not know that function. [In Chapter 5, we will examine some models that do give us the map function directly. However, for systems described by differential equations in state spaces of three (or more) dimensions, it is in general impossible to find the map functions.]

Stability of Limit Cycles

As we saw in two-dimensional systems, if the fixed point is to be stable and have trajectories in its neighborhood attracted to it, then the *absolute value* of each multiplier must be less than 1. [In state spaces with three or more dimensions, we can have $M < 0$; so the stability criterion is formulated using the absolute value of the multipliers.]

The types of limit cycles are

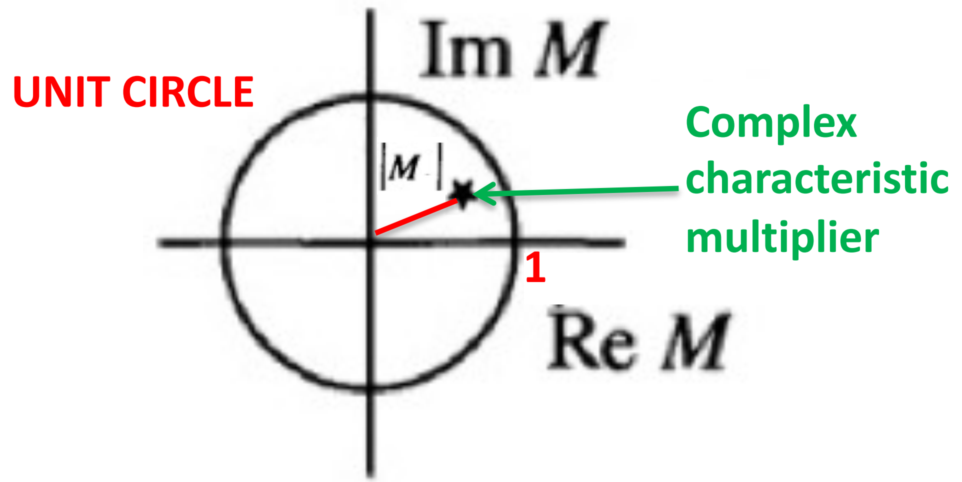
I. ***Stable limit cycle*** (node for the Poincaré map)
II. ***Repelling limit cycle*** (repellor for the Poincaré map)
III. ***Saddle cycle*** (saddle point for the Poincaré map)

Table 4.2 lists the categories of characteristic multipliers, the associated Poincaré plane fixed points and the corresponding limit cycles for three-dimensional state spaces. (Compare this table to Table 3.4 for limit cycles in two-dimensional state spaces.)

Table 4.2
Characteristic Multipliers for Poincaré Sections
of Three-Dimensional State Spaces

Type of Fixed Point	Characteristic Multiplier	Corresponding Cycle
Node	$\lvert M_1 \rvert, \lvert M_2 \rvert < 1$	Limit Cycle
Repellor	$\lvert M_1 \rvert, \lvert M_2 \rvert > 1$	Repelling Cycle
Saddle	$\lvert M_1 \rvert < 1, \lvert M_2 \rvert > 1$	Saddle Cycle

Of course, the characteristic multipliers could also be complex numbers. Just as we saw for fixed points in a two-dimensional state space, the complex multipliers will form a complex-conjugate pair. In more graphic terms, the successive Poincaré intersection points associated with complex-valued multipliers rotate around the limit cycle intersection point as they approach or diverge from that point. Mathematically, the condition for stability is still the same: the absolute value of both multipliers must be less than 1 for a stable limit cycle. In terms of the corresponding Argand diagram (complex mathematical plane), both characteristic values must lie within a circle of unit radius (called the ***unit circle***) for a stable limit cycle. See Fig. 4.7. As a control parameter is changed the values of the characteristic multipliers can change. If at least one of the characteristic multipliers crosses the unit circle, a bifurcation occurs. Some of these bifurcations will be discussed in the latter part of this chapter.



Stable Limit Cycles

$|M_1|, |M_2| < 1$

Repelling Limit Cycles

$|M_1|, |M_2| > 1$

Saddle Limit Cycles

$|M_1| < 1, |M_2| > 1$

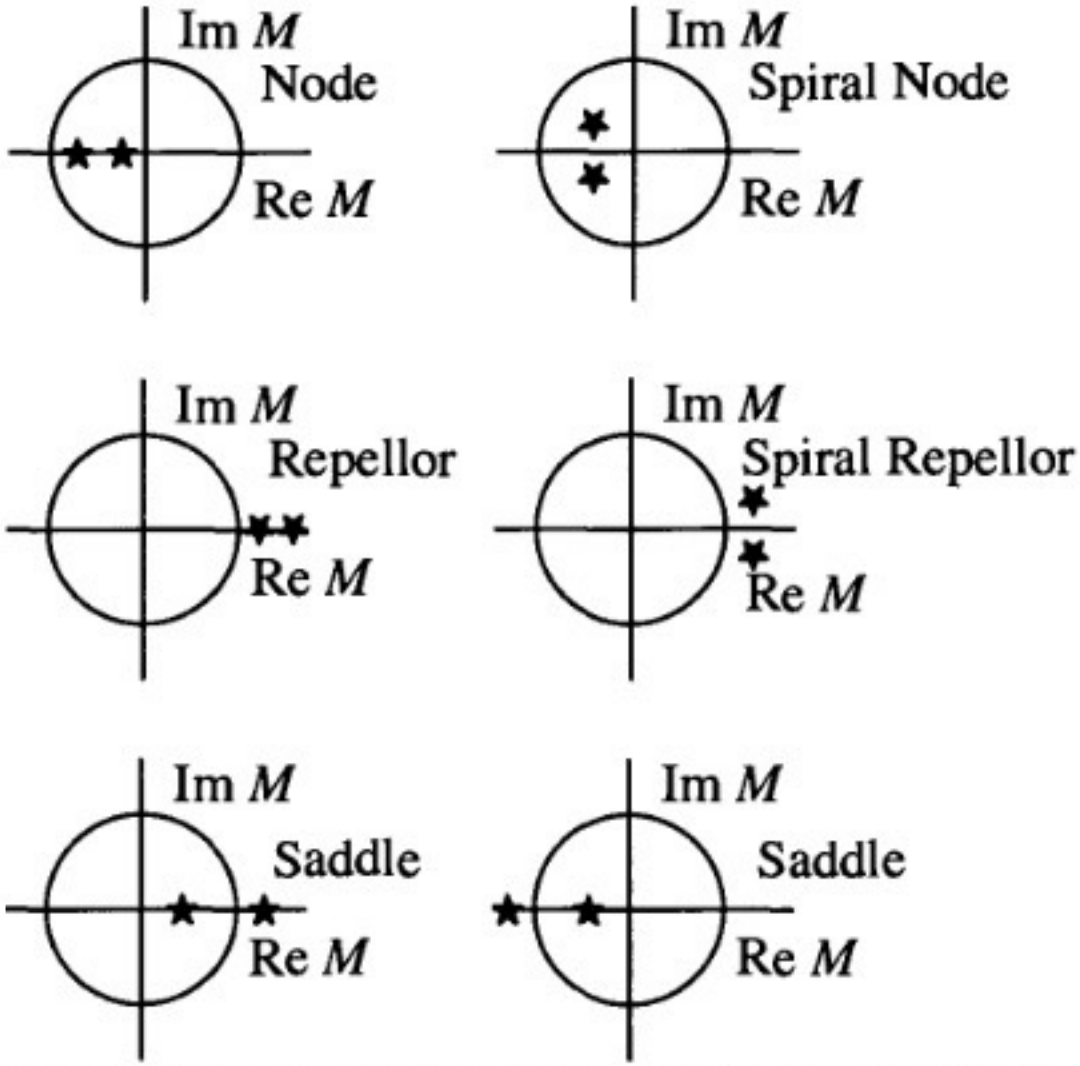
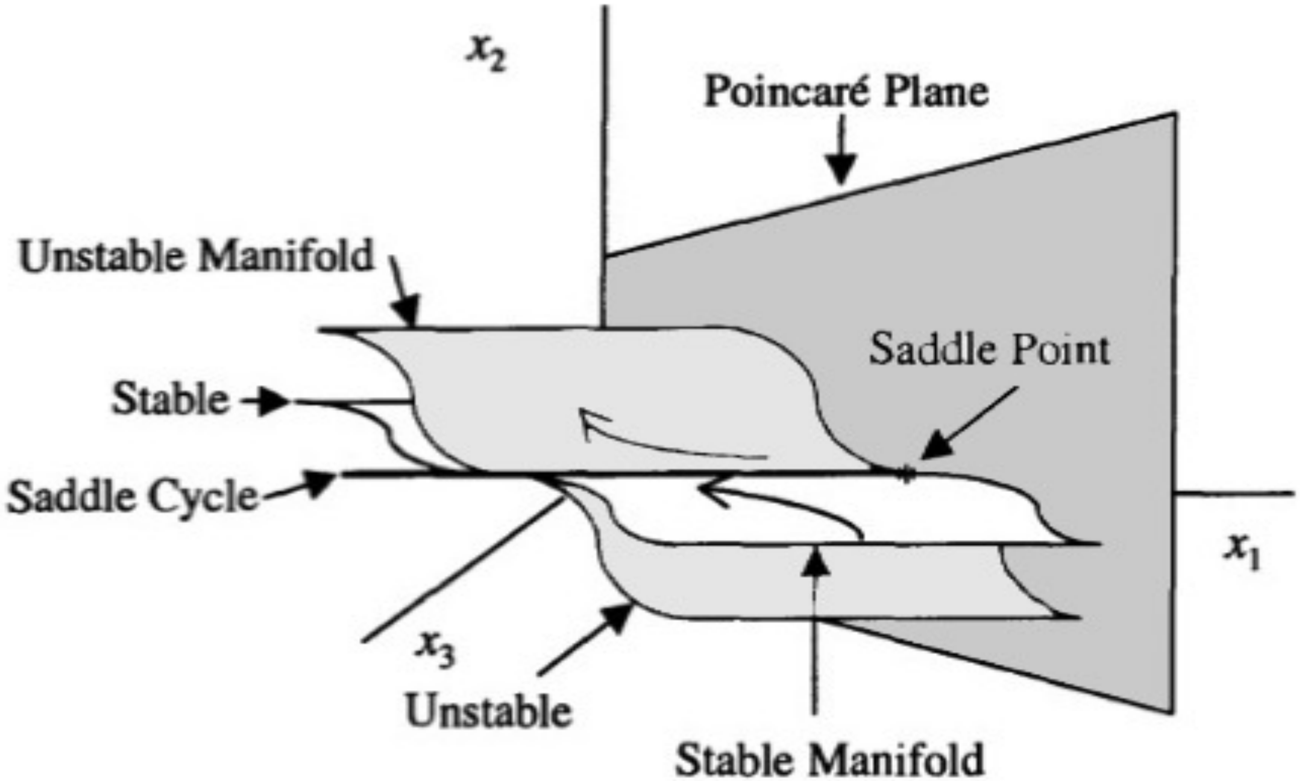


Fig. 4.7. Characteristic multipliers in the complex plane. If both multipliers lie within a circle of unit radius (the unit circle), then the corresponding limit cycle is stable. If one (or both) of the multipliers lies outside the unit circle, then the limit cycle is unstable.



Saddle Limit Cycles

$|M_1| < 1, |M_2| > 1$

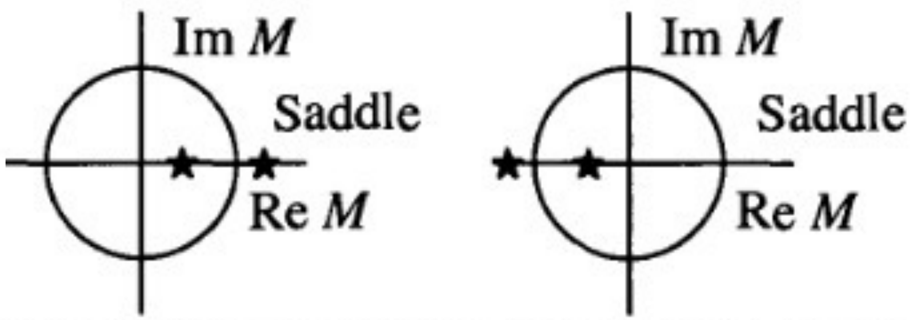


Fig. 4.7. Characteristic multipliers in the complex plane. If both multipliers lie within a circle of unit radius (the unit circle), then the corresponding limit cycle is stable. If one (or both) of the multipliers lies outside the unit circle, then the limit cycle is unstable.

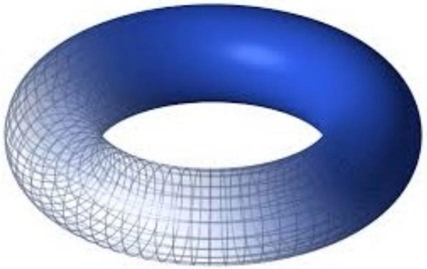
4.7 Quasi-Periodic Behavior (dim = 2)

For a three-dimensional state space, a new type of motion can occur, a type of motion not possible in one- or two-dimensional state spaces. This new type of motion is called *quasi-periodic* because it has two different frequencies associated with it; that is, it can be analyzed into two independent, periodic motions. For quasi-periodic motion, the trajectories are constrained to the surface of a torus in the three-dimensional state space. A mathematical description of this kind of motion is given by:

NOTA BENE: NON SONO EQUAZIONI DIFFERENZIALI!

Equazioni della traiettoria di un punto (x_1(t), x_2(t) e x_3(t)) sull'attrattore quasiperiodico

$$
\begin{aligned}
x_1 &= (R + r\sin\omega_r t)\cos\omega_R t \\
x_2 &= r\cos\omega_r t \\
x_3 &= (R + r\sin\omega_r t)\sin\omega_R t
\end{aligned}
\tag{4.7-1}
$$



where the two angular frequencies are denoted by ω_R and ω_r. Geometrically, Eqs. (4.7-1) describe motion on the surface of a torus (with the center of the torus at the origin), whose large radius is R and whose cross-sectional radius is r. In general the torus (or doughnut-shape or the shape of the inner tube of a bicycle tire) will look something like Fig. 4.8. The frequency ω_R corresponds to the rate of rotation around the large circumference with a period $T_R = 2\pi/\omega_R$, while the frequency ω_r corresponds to the rate of rotation about the cross section with $T_r = 2\pi/\omega_r$. A general torus might have elliptical cross sections, but the ellipses can be made into circles by suitably rescaling the coordinate axes.

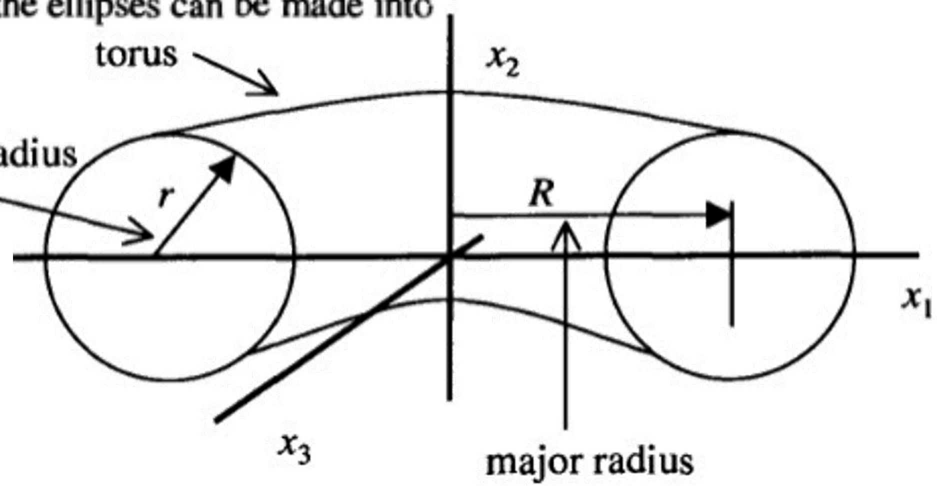
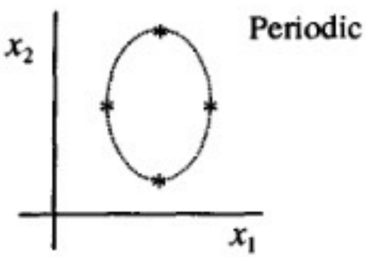


Fig. 4.8. Quasi-periodic trajectories roam over the surface of a torus in three-dimensional state space. Illustrated here is the special case of a torus with circular cross sections. r is the minor radius of the cross section. R is the major radius of the torus. A periodic trajectory on the surface of the torus would close on itself. On the right, a perspective view of a torus and a Poincaré plane.

The Poincaré section for this motion is generated by using a Poincaré plane that cuts through the torus. What the pattern of Poincaré map points looks like depends on the numerical relationship between the two frequencies as illustrated in Fig. 4.9. If the ratio of the two frequencies can be expressed as the ratio of two integers (that is, as a "rational fraction," 14/17, for example), then the Poincaré section will consist of a finite number of points. This type of motion is often called ***frequency-locked*** motion because one of the frequencies is locked, often over a finite control parameter range, so that an integer multiple of one frequency is equal to another integer multiple of the other. (The terms ***phase-locking*** and ***mode-locking*** are also used to describe this behavior.)

$$\frac{\omega_R}{\omega_r}$$

Winding Number razionale



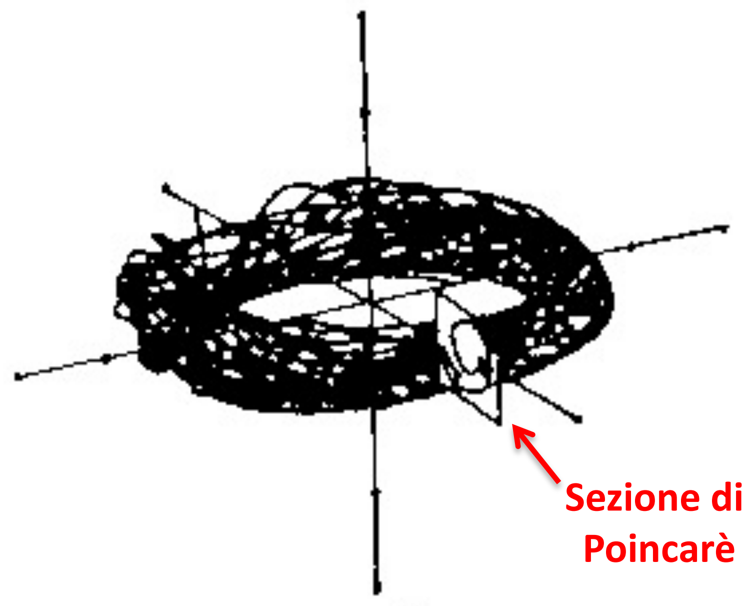
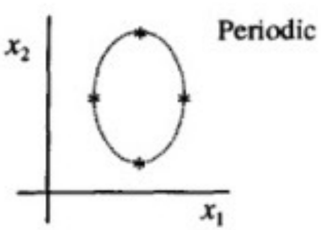


Fig. 4.9. A Poincaré section intersects a torus in three–dimensional state space. The diagram on the upper left shows the Poincaré map points for a two-frequency periodic system with a rational ratio of frequencies. The intersection points are indicated by asterisks. The diagram on the lower left is for quasi-periodic behavior. The ratio of frequencies is irrational, and eventually the intersection points fill in a curve (sometimes called a "drift ring") in the Poincaré plane.

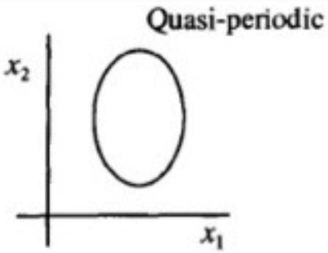
If the ratio of frequencies cannot be expressed as a ratio of integers, then the ratio is called "irrational" (in the mathematical, not the psychological sense). For the irrational case, the Poincaré map points will eventually fill in a continuous curve in the Poincaré plane, and the motion is said to be ***quasi-periodic*** because the motion never exactly repeats itself. (Russian mathematicians call this ***conditionally periodic***. See, for example, [Arnold, 1983]. The term ***almost periodic*** is also used in the mathematical literature.)

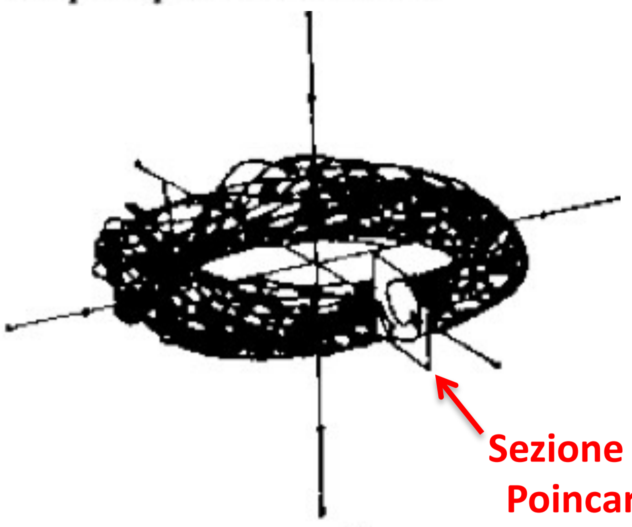
In the quasi-periodic case the motion, strictly speaking, never exactly repeats itself (hence, the modifier *quasi*), but the motion is not chaotic; it is composed of two (or more) periodic components, whose presence could be made known by measuring the frequency spectrum (Fourier power spectrum) of the motion. We should point out that detecting the difference between quasi-periodic motion and motion with a rational ratio of frequencies, when the integers are large, is a delicate question. Whether a given experiment can distinguish the two cases depends on the resolution of the experimental equipment. As we shall see later, the behavior of the system can switch abruptly back and forth between the two cases as a parameter of the system is varied. The important point is that the attractor for the system is a two-dimensional surface of the torus for quasi-periodic behavior.



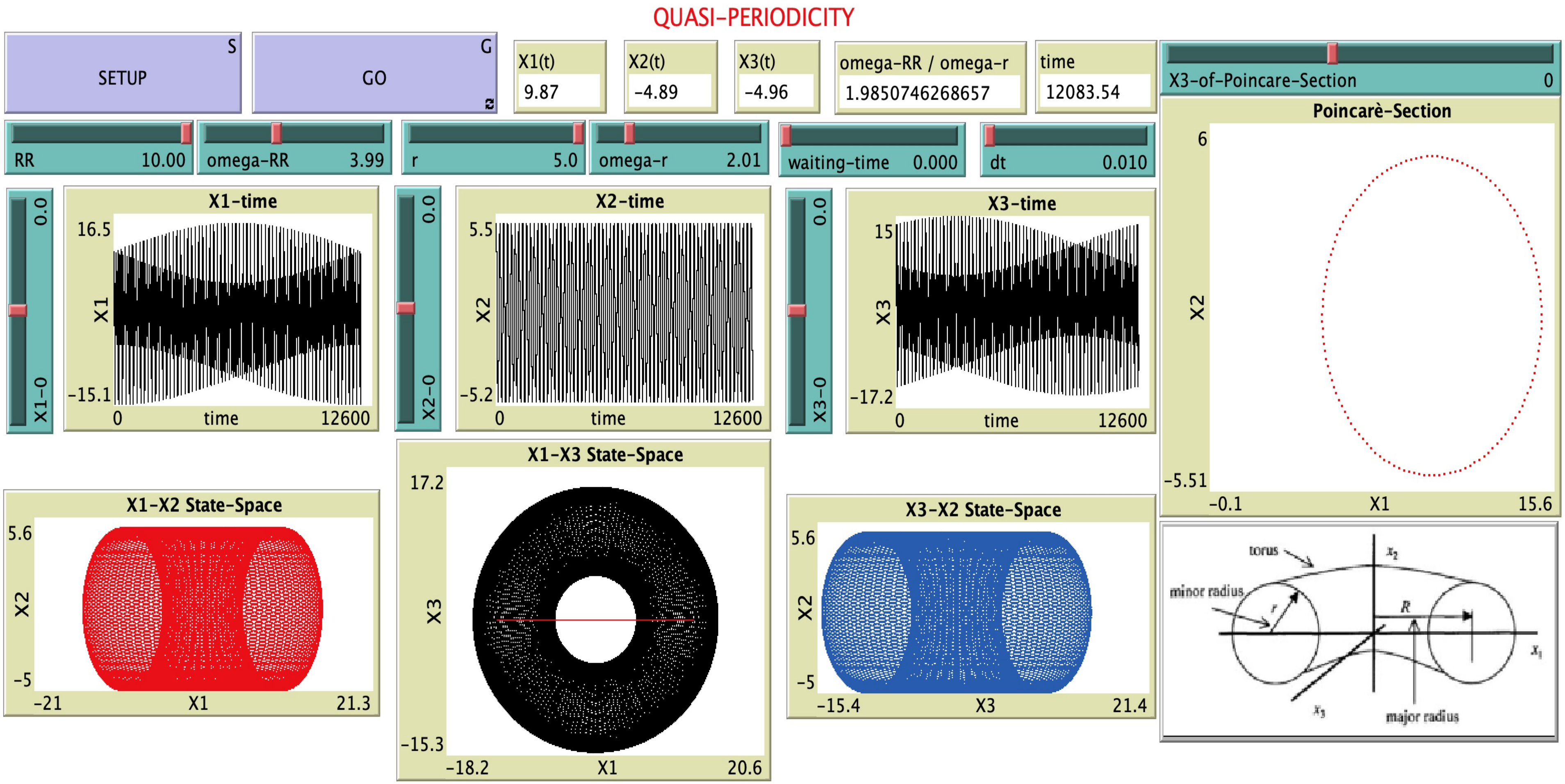
Winding Number irrazionale

$$\frac{\omega_R}{\omega_r}$$





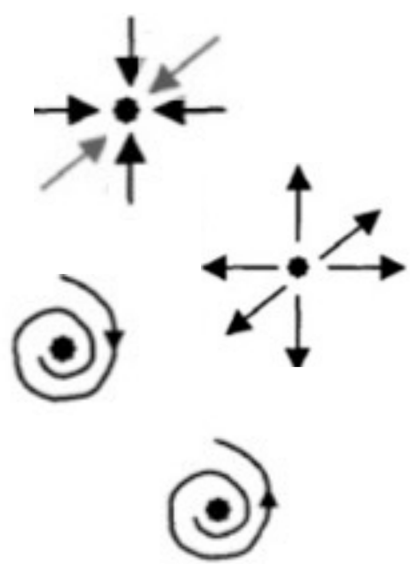
quasi-periodicity.nlogo



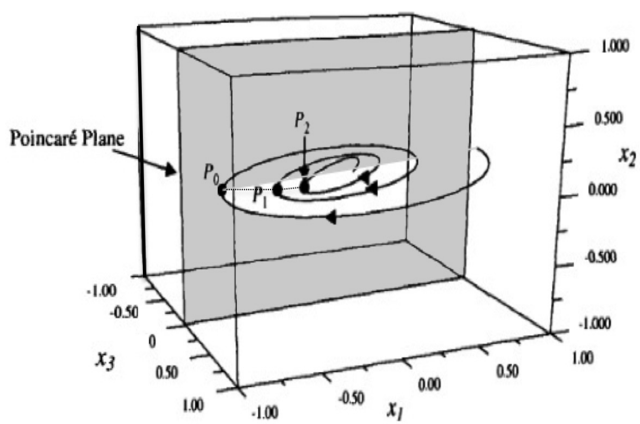
We have now seen the full panoply of regular (nonchaotic) attractors: fixed points (dimension 0), limit cycles (dimension 1), and quasi-periodic attractors (dimension 2 or more). We are ready to begin the discussion of how these attractors can change into chaotic attractors.

We will give only a brief discussion of the period-doubling, quasi-periodic, and intermittency routes. These will be discussed in detail in Chapters 5, 6, and 7, respectively. A discussion of crises will be found in Chapter 7. As we shall see, the chaotic transient route is more complicated to describe because it requires a knowledge of what trajectories are doing over a range of state space. We can no longer focus our attention locally on just a single fixed point or limit cycle.

fixed points (dim.0)



limit cycles (dim.1)



quasiperiodic attractors (dim.2)