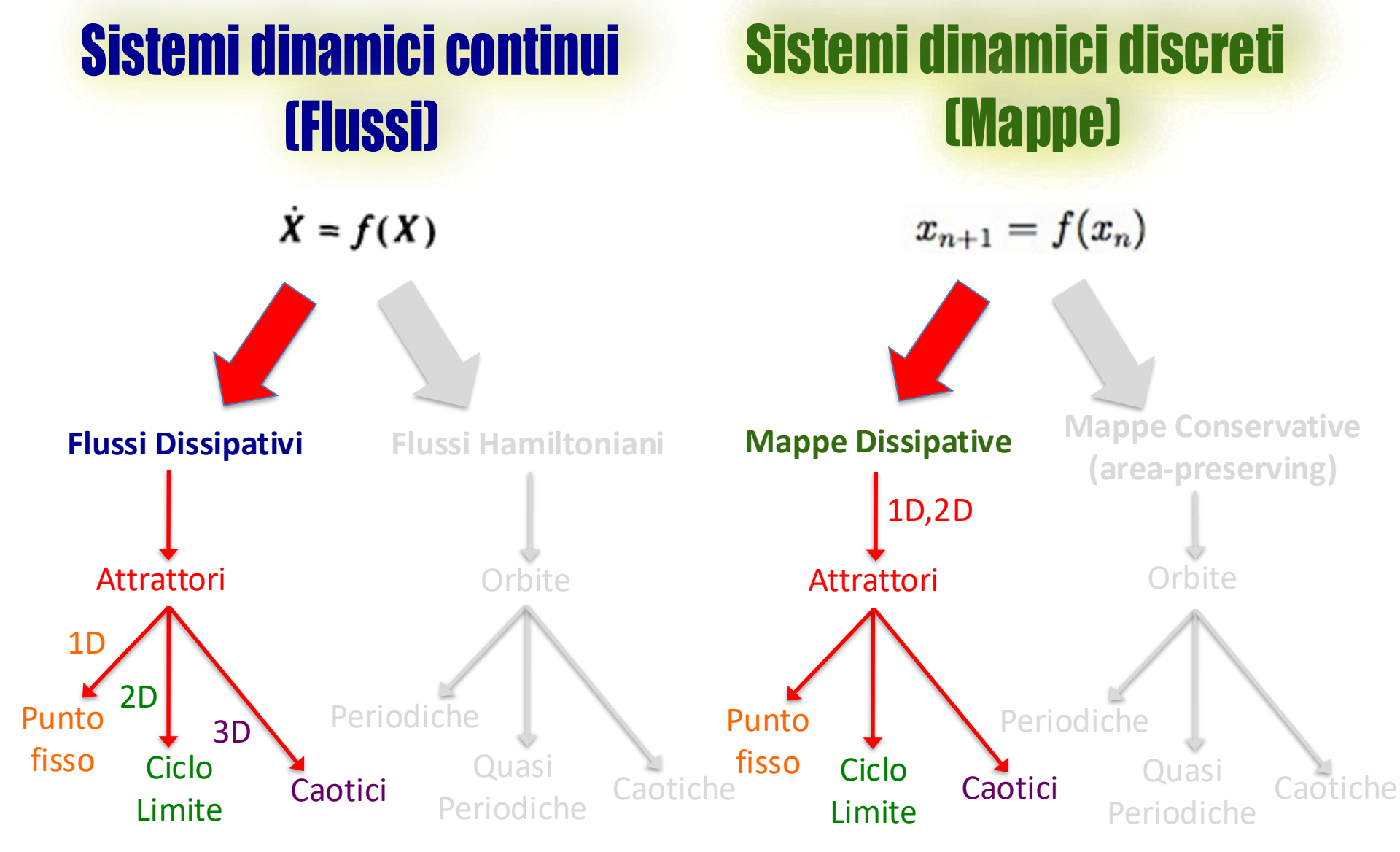
Classificazione dei Sistemi Dinamici



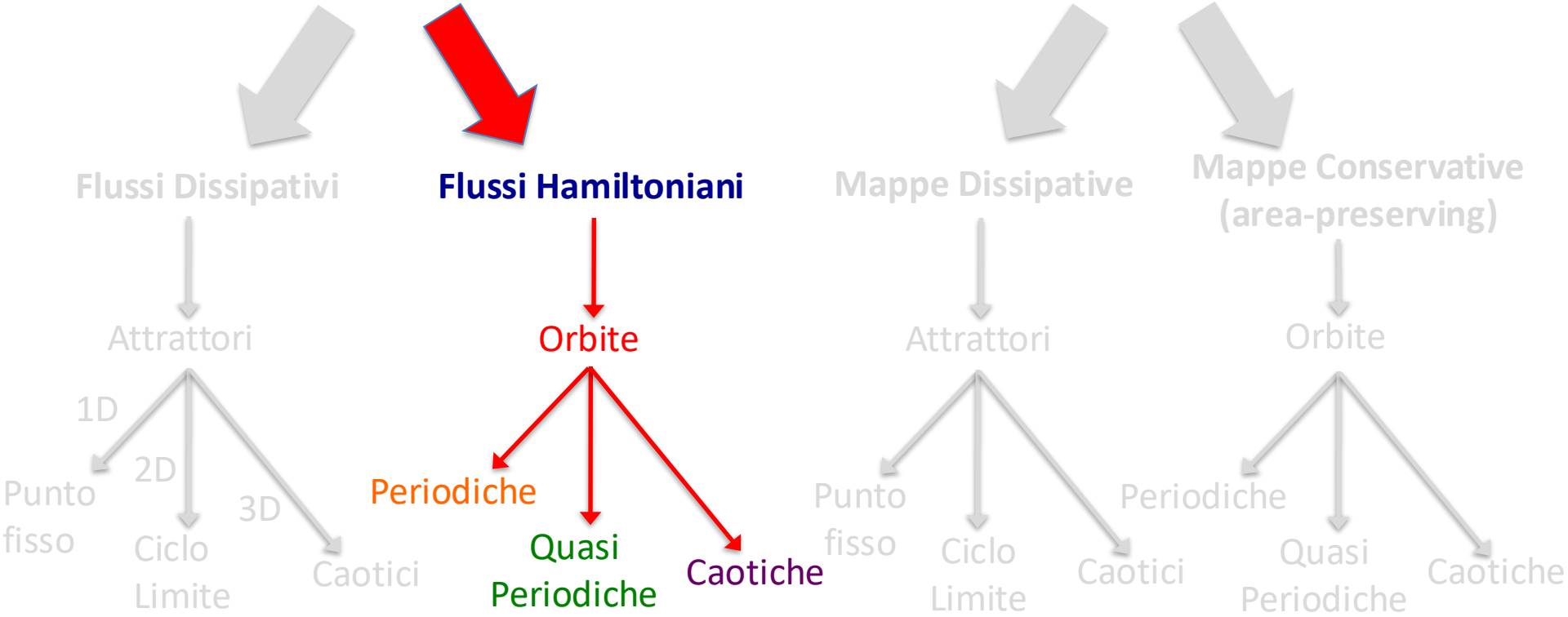
Classificazione dei Sistemi Dinamici

Sistemi dinamici continui (Flussi)

$$\dot{X} = f(X)$$

Sistemi dinamici discreti (Mappe)

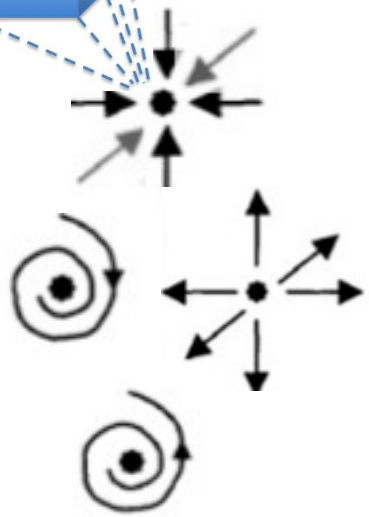
$$x_{n+1} = f(x_n)$$

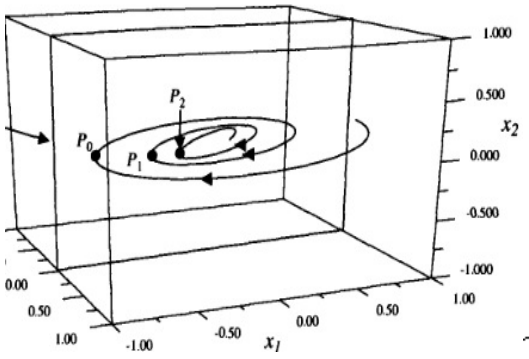


Hamiltonian Systems

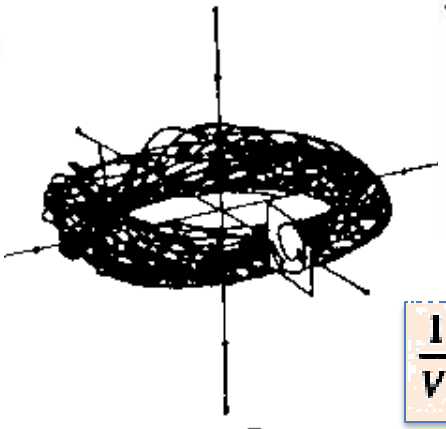
8.1 Introduction

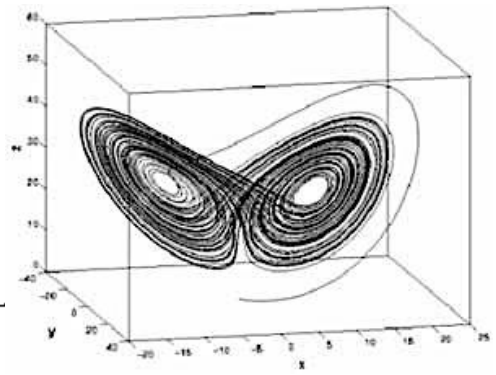
In our discussions of nonlinear dynamics up to this point, we have dealt only with dissipative systems. The crucial feature of a dissipative system from the state space point of view is the "collapse" of a volume of initial conditions in state space. For most purposes, we can focus our attention on the attractor (or attractors, in general) in state space—those "areas" to which trajectories from a range of initial conditions are attracted. That is, we need consider only the attractors to understand the long-term dynamics of the system.





SISTEMI DISSIPATIVI





Il volume V prima o poi collassa su un attrattore

$$\frac{1}{V}\frac{dV}{dt} = \sum_{i=1}^{N}\frac{\partial f_i}{\partial x_i} \equiv div(f) < 0$$

Hamiltonian Systems

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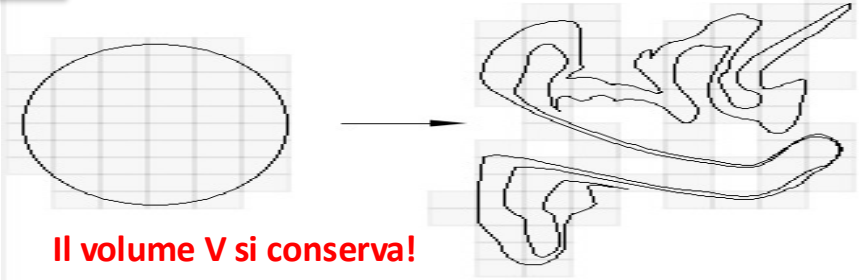
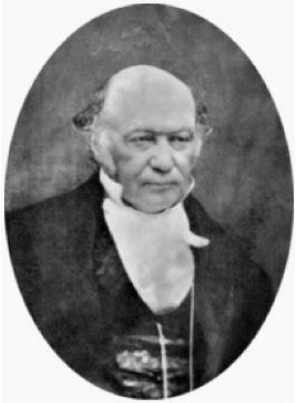
What happens if the amount of dissipation becomes smaller and smaller? In that case the system obviously takes longer and longer for trajectories that start away from the attractor to approach the attractor; it takes longer for a volume of initial conditions to collapse onto the attractor. In the limit in which there is no dissipation at all, we would expect that a volume of initial conditions would remain constant for all time and that there would exist no attractors for the trajectories.

SISTEMI HAMILTONIANI



Il volume V si conserva!

$$\frac{1}{V}\frac{dV}{dt} = \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i} \equiv div(f) = 0$$


William Rowan Hamilton (1805–1865)

Systems (or models) with no dissipation are called ***conservative systems***, or equivalently, ***Hamiltonian systems***. The term conservative means that certain physical properties of the system (such as the total mechanical energy, the angular momentum, etc.) remain constant in time. We say that these quantities are ***conserved*** quantities. If the system starts with a certain amount of energy, then that amount of energy stays the same over time. The name Hamiltonian is applied to these systems because their time evolution can be described by the so-called Hamilton's equations (after Sir William Hamilton, 1805–1865, a noted Scottish mathematician). We shall discuss these equations in the next section.

Do conservative systems occur in nature? In principle, the answer to this depends on our "level of description." Since we know that the total energy of an isolated system is conserved, though the energy may change form, we might conclude that Hamiltonian models are the only appropriate models. However, in practice, this full description is often too complex, and we instead focus our attention on one particular subsystem; the remaining part of the system acts as a source or sink of energy (i.e., as a source of dissipation). In that case, a dissipative model is appropriate.

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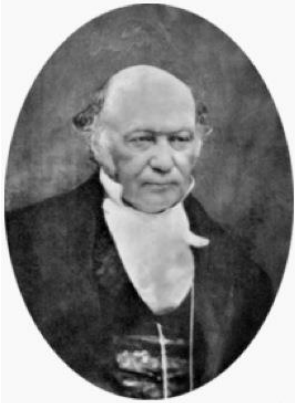
SISTEMA ISOLATO
CONSERVATIVO



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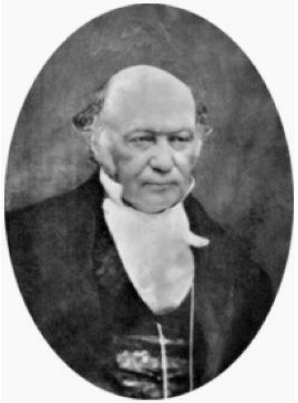
TERMOSTATO

SISTEMA CHIUSO

DISSIPATIVO



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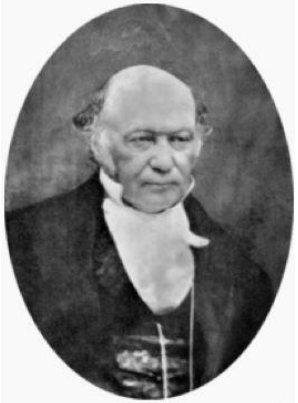
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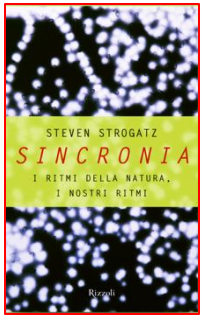
TERMOSTATO

SISTEMA
≈ ISOLATO
≈ CONSERVATIVO

In practice, many real systems are nearly conservative. The most famous (almost) conservative system is the solar system. In fact, it was a consideration of the dynamics of the solar system that led Poincaré to introduce many of the methods already described for dealing with nonlinear dynamics. Over the time periods of concern, which for the solar system are millions and billions of years, we can neglect most aspects of dissipation. There are, however, dissipation effects in the solar system such as tidal forces, which act on planets and moons, and the drag effects of the "solar wind," streams of particles emitted by the Sun. For example, dissipative tidal forces are responsible for the locking of the Moon's rotation rate to its orbital period so that the same side of the Moon always faces the Earth, as mentioned in Chapter 6. To a high degree of approximation, however, these dissipative effects can be neglected if we limit ourselves to time periods of a few million years. Based on these considerations, we can model the dynamics of the solar system with a dissipation-free, conservative (Hamiltonian) model.




Vita pensata

HOME AUTORI EDITORIALI NEES RECENSIONI SCRITTURA CREATIVA

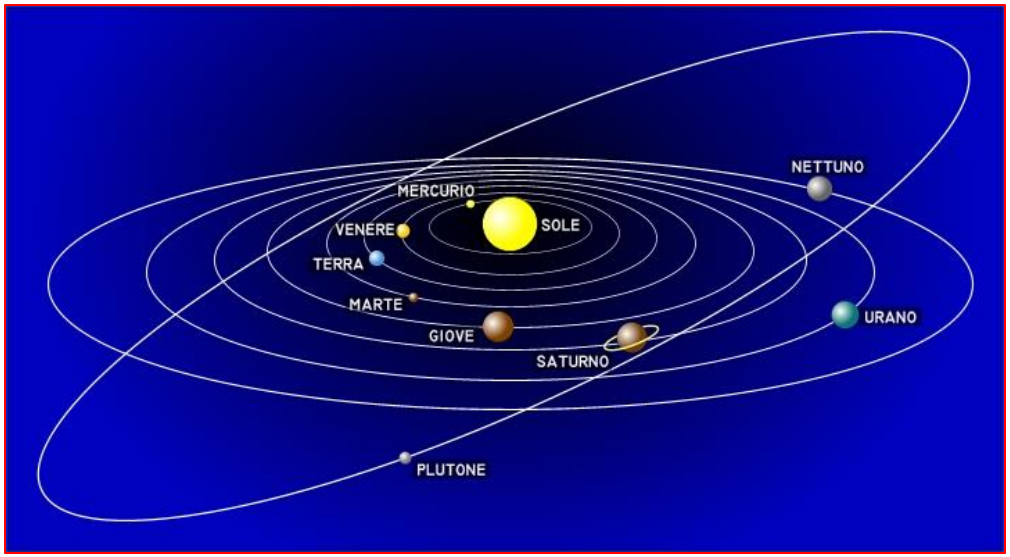
(Tre) corpi al margine del caos

Di: **Alessandro Pluchino**

4 Gennaio 2022

Come è noto, il Tre è spesso considerato il *numero perfetto* da diversi p
(due) e del dispari (uno); dal punto di vista *esoterico* è il simbolo dell
rappresenta la perfezione divina (si pensi alla Trinità del Cristianesimo o a
rappresenta anche la *soglia dell'imperfezione*, il numero magico che ha co
regione oggi conosciuta come "Margine del Caos", spalancando così le
rivoluzione concettuale è partita riguardava un problema di corpi. Per la pr

Tutto cominciò la notte
di Magny, nella Borgogn
miniera, ustionando e u
lavoro a quell'ora. Fu s
delle indagini a permet
lampada perforata accic
un'atmosfera ricca di
avrebbe portato alla co
Jules-Henri Poincaré, co
fisici di fine Ottocento
tempo!) e che è consic
assoluto della moderna
per raccontarvi.



https://www.vitapensata.eu/2022/01/04/tre-corpi-al-margine-del-caos/

Hamiltonian models are also important in the study of proton beams in high-energy particle accelerations, in quantum mechanics (more on this in Chapter 12), and as a branch of applied mathematics, for which there is now a vast literature. In this chapter, we will describe how chaos appears in Hamiltonian systems, and we will severely limit the amount of mathematical detail so that we can focus on how Hamiltonian systems differ from dissipative systems in some respects but are similar in others. By looking at a model with a variable amount of dissipation, we shall see how the two types of systems are connected. Most of the important theoretical results will simply be stated with only a sketch of a plausibility argument. The goal is to give you an intuitive picture of the rather complex behavior exhibited by Hamiltonian systems. Once the overall picture is in hand, the mathematically inclined reader can use the references at the end of the chapter for more detailed treatments.

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We will first introduce some of the basic notions of Hamiltonian systems including the state-space description. We will then discuss an important, but limited, subclass of Hamiltonian systems—those called ***integrable***. However, integrable systems cannot show chaotic behavior; therefore, we must explore what happens when a Hamiltonian system becomes nonintegrable. The chapter concludes with a brief description of some applications.

8.2 Hamilton's Equations and the Hamiltonian

Although we shall not make much direct use of Hamilton's equations, it will be helpful to introduce them briefly, both for the following discussion and for the chance to become familiar with some of the specialized jargon used in the study of Hamiltonian systems. In the Hamilton formulation of classical (Newtonian) mechanics, the time evolution of a system is described in terms of a set of dynamical variables, which give the positions (coordinates) and the momenta of the particles of the system. Traditionally, the coordinates are indicated with the symbols q_i and the momenta by p_i. The subscript notation is used to pick out a particular particle and a particular component of the position vector and momentum vector for that particle. If the system consists of N point particles, each of which has three components for its position vector and three components for its momentum vector, the subscript i will run from 1 to $3N$. For example, we might have $q_1 = (\vec{r}_1)_x$ and $p_1 = (\vec{p}_1)_x$. Here, q_1 represents the x component of the position vector for particle number 1, and p_1 the corresponding x component of the particle's momentum vector. Each pair q_i, p_i corresponds to a "degree-of-freedom" for the Hamiltonian system. (Recall the discussion in Section 3.2 about different uses of the term degree-of-freedom.)

.....

*Coordinate Part.*1 $(q_1, q_2, q_3, p_1, p_2, p_3)$ $\square$ $(r_{1x}, r_{1y}, r_{1z}, p_{1x}, p_{1y}, p_{1z})$

*Coordinate Part.*2 $(q_4, q_5, q_6, p_4, p_5, p_6)$

Coordinate Part.N $(q_{3N-2}, q_{3N-1}, q_{3N}, p_{3N-2}, p_{3N-1}, p_{3N})$

The evolution of the Hamiltonian system is completely described if the time dependence of the qs and ps is known. That is, if we know $q_i(t)$ and $p_i(t)$ for all t and for all i, then we know everything there is to know about the time behavior of the system. In the Hamilton formulation, the time-dependence of the qs and ps is determined by solutions of Hamilton's equations, which are written in terms of the derivatives of the ***Hamiltonian function*** (or just ***Hamiltonian***, for short) $H(q,p)$, where the unadorned symbols q and p mean that H depends, in general, on all the q_i and p_i. For the simplest cases, the Hamiltonian is just the total mechanical energy (kinetic energy plus potential energy) of the system, written as a function of the qs and ps. In any case, Hamilton's equations are a set of $2N$ coupled differential equations (for a system of N degrees of freedom)

Hamiltoniana

$H(q,p)$ □

$E_{tot} = K(p) + U(q)$

$$\begin{cases} \dfrac{dq_i}{dt} = \dfrac{\partial H(q,p)}{\partial p_i} \longrightarrow q_i(t) \\ \dfrac{dp_i}{dt} = -\dfrac{\partial H(q,p)}{\partial q_i} \quad i = 1,\ldots,N \longrightarrow p_i(t) \end{cases} \tag{8.2-1}$$

Exercise 8.2-1. Suppose a single particle with a mass m is constrained to move along the x axis. Its Hamiltonian $H = p_x^2/2m + U(q_x)$ is the sum of a kinetic energy term and a potential energy U. Show that Hamilton's equations are equivalent to Newton's Laws of Motion for the system. Hint: The x component of the force acting on the system is given by $F_x = -dU/dx$, the negative gradient (in the x direction) of the potential energy function.

Note that Hamilton's equations are similar in form to the standard first-order differential equations we have been using to describe dynamics in state space for a variety of systems. The similarity can be made more obvious by identifying the state space variables $x_1 = q_1, x_2 = p_1, x_3 = q_2$, and so on. For a Hamiltonian system, the functions (analogous to the *f*s in our previous treatment) that give the time dependence of the state space variables can be written as (partial) derivatives of some common function, namely, the Hamiltonian. As we shall see in the next section, that crucial feature embodies the special nature of Hamiltonian systems. The special linkage between the *q*s and *p*s and the partial derivatives of the Hamiltonian function give Hamiltonian mechanics a special mathematical form called a ***symplectic*** structure, which can be exploited to give elegant proofs of many features of the time behavior (see, for example, [Goldstein, 1980]).

Hamilton's Equations

$$\begin{cases} \dfrac{dq_i}{dt} = \dfrac{\partial H(q,p)}{\partial p_i} \\ \dfrac{dp_i}{dt} = -\dfrac{\partial H(q,p)}{\partial q_i} \end{cases} \quad i = 1,\ldots,N$$

$$\begin{cases} \dot{q}_i = f_i(q_i, p_i) \\ \dot{p}_i = g_i(q_i, p_i) \end{cases} \quad i = 1\ldots N$$

Sistema Hamiltoniano ad N gradi di libertà (2N dimensioni)

$$f_i(q_i, p_i) = \frac{\partial H(q_i, p_i)}{\partial p_i}$$

$$g_i(q_i, p_i) = -\frac{\partial H(q_i, p_i)}{\partial q_i}$$

An important consequence follows from Hamilton's equations: The value of the Hamiltonian itself represents a conserved quantity; it does not vary in time. We prove this by using the chain rule of differentiation:

$$\frac{dH(q,p)}{dt} = \sum_i \left\{ \frac{\partial H}{\partial p_i}\frac{dp_i}{dt} + \frac{\partial H}{\partial q_i}\frac{dq_i}{dt} \right\} + \frac{\partial H}{\partial t} \qquad (8.2\text{-}2)$$

The terms inside the braces of Eq. (8.2-2) tell us how H depends on time because H depends on the qs and ps and the qs and ps (in general) depend on time. This part describes the so-called ***implicit*** time-dependence of H. The last term in Eq. (8.2-2) tells us how H depends on time if the time variable appears ***explicitly*** in H. Explicit time dependence occurs if the system is subject to an externally applied time-dependent force, for example. We will not consider such cases; therefore, we will assume that the last term is 0.

If we now use Hamilton's equations (8.2-1) in Eq. (8.2-2), we find for each term in the sum

$$\frac{dq_i}{dt} = \frac{\partial H(q,p)}{\partial p_i}$$
$$\frac{dp_i}{dt} = -\frac{\partial H(q,p)}{\partial q_i}$$

$$\frac{\partial H}{\partial p_i}\left(-\frac{\partial H}{\partial q_i} \right) + \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} = 0 \qquad (8.2\text{-}3)$$

So, we see that the time derivative of H is 0 (if H does not depend explicitly on time). Hence, H represents a conserved quantity. If H represents the total energy of the system (as it usually does), then we say that the total energy is conserved for a Hamiltonian system. Alternatively, we say that the total energy is a "constant of the motion."

8.3 Phase Space

We again find that a geometric state space description is useful, if not essential, for understanding how chaos occurs in Hamiltonian systems. State space for a Hamiltonian system is traditionally called ***phase space***, and the axes of phase space give the values of the *q*s and *p*s. Hence, if we have *N* degrees of freedom (in the Hamiltonian sense of that phrase), we have *N* pairs of *q*s and *p*s, and the phase space will have 2*N* dimensions. Thus, for a Hamiltonian system (with no explicit time dependence in *H*), the phase space always has an even number of dimensions. Even for simple cases we will have difficulty visualizing these multi-dimensional phase spaces; therefore, we can anticipate using projections and Poincaré sections to simplify the description.

$$\begin{cases} \dfrac{dq_i}{dt} = \dfrac{\partial H(q,p)}{\partial p_i} \\ \dfrac{dp_i}{dt} = -\dfrac{\partial H(q,p)}{\partial q_i} \end{cases} \quad i = 1,\ldots,N$$

N=1

$$\begin{cases} \dot{q}_1 = \dot{x}_1 = f_1(x_1,x_2) \\ \dot{p}_1 = \dot{x}_2 = f_2(x_1,x_2) \end{cases}$$

Sistema Hamiltoniano ad 1 grado di libertà (Phase space a 2N=2 dimensioni, non si può avere caos)

N=2

$$\begin{cases} \dot{q}_1 = \dot{x}_1 = f_1(x_1,x_2,x_3,x_4) \\ \dot{p}_1 = \dot{x}_2 = f_2(x_1,x_2,x_3,x_4) \\ \dot{q}_2 = \dot{x}_3 = f_3(x_1,x_2,x_3,x_4) \\ \dot{p}_2 = \dot{x}_4 = f_4(x_1,x_2,x_3,x_4) \end{cases}$$

Sistema Hamiltoniano a 2 gradi di libertà (Phase space a 2N=4 dimensioni, si può avere caos!)

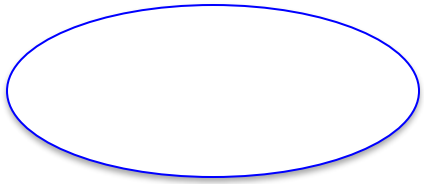
Since the Hamiltonian function value (usually the energy of the system) is a constant of the motion, a trajectory for a Hamiltonian system cannot go just anywhere in phase space. It can go only to regions of (q, p) space that have the same energy value as the initial point of the trajectory. Thus, we say that trajectories in phase space are confined to a $2N - 1$ dimensional ***constant energy surface***. (Of course, this "surface" may be a multidimensional geometric object in general.)

Esempi:

Sistema Hamiltoniano ad 1 grado di libertà (2N=2 dimensioni)

2N-1=1 dim. →

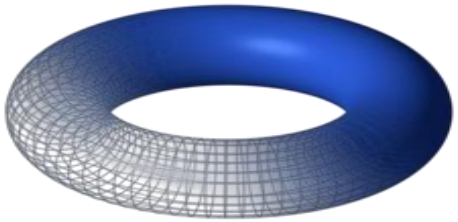
La traiettoria è confinata lungo una linea chiusa 1D (toro unidimensionale), luogo dei punti dello spazio delle fasi ad energia costante



Sistema Hamiltoniano a 2 gradi di libertà (2N=4 dimensioni)

2N-1=3 dim. →

La traiettoria è confinata all'interno del volume 3D di un toro, ipersuperficie ad energia costante

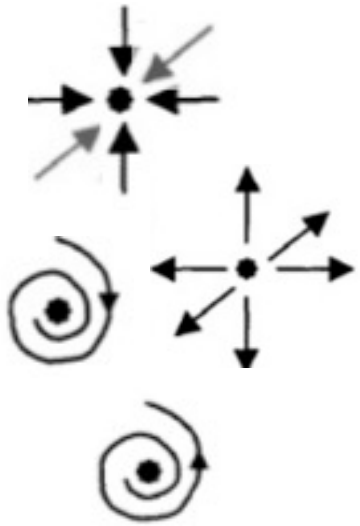


REMINDER:

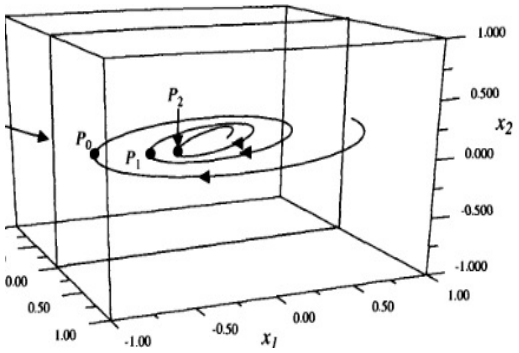
Flussi dissipativi

$$\frac{1}{V}\frac{dV}{dt} = \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i} \equiv div(f) \quad < 0$$

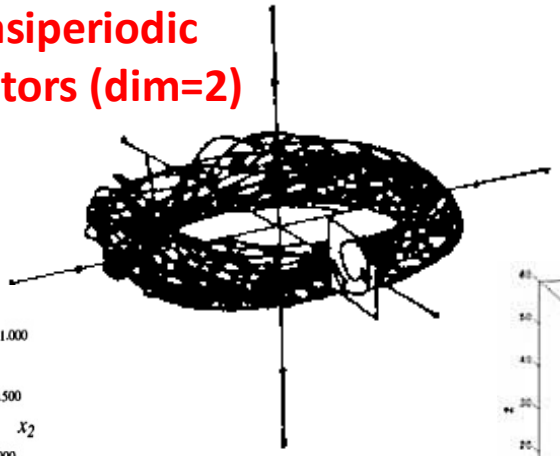
fixed points (dim=0)



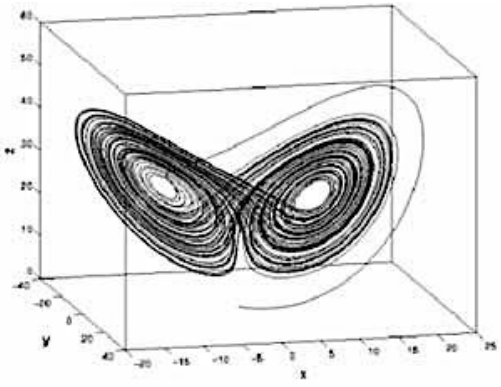
limit cycles (dim=1)



quasiperiodic attractors (dim=2)



chaotic attractors (2<dim<3)



Conservazione del volume nello Spazio delle Fasi

Using techniques similar to those employed for dissipative systems, we can now show that a volume in phase space occupied by a set of initial conditions remains the same as the Hamiltonian system evolves. In Chapter 3, we established that the time-dependence of a small volume V occupied by a set of initial conditions in state space is given by

$$\frac{1}{V}\frac{dV}{dt} = \sum_i \frac{\partial f_i}{\partial x_i} \tag{8.3-1}$$

To show that this time derivative is 0 for a Hamiltonian system, we first need to translate Eq. (8.3-1) into the language of Hamiltonian dynamics. It will be useful in our proof to identify, as we did earlier, x_1 with q_1, x_2 with the corresponding p_1, x_3 with q_2, x_4 with p_2, and so on. We also recall that the time dependence of the xs is given by an equation of the form

$$\dot{x}_i = f_i(x_1, x_2, \ldots) \tag{8.3-2}$$

If we have $2N$ dimensions in the phase space, the index i will run from 1 to $2N$.

$$\dot{q}_1 = \dot{x}_1 = f_1(x_1, \ldots) = \frac{\partial H}{\partial p_1} \tag{8.3-3}$$

$$\dot{p}_1 = \dot{x}_2 = f_2(x_1, \ldots) = -\frac{\partial H}{\partial q_1} \tag{8.3-4}$$

where the last equality in the two previous equations follows from Hamilton's equations.

Let us now examine the terms that appear in the sum on the right-hand side of Eq. (8.3-1). In particular let us look at the first two terms and insert the results of Eqs. (8.3-3) and (8.3-4):

$$\frac{1}{V}\frac{dV}{dt} = \sum_i \frac{\partial f_i}{\partial x_i}$$

$$\dot{q}_1 = \dot{x}_1 = f_1(x_1,\ldots) = \frac{\partial H}{\partial p_1}$$

$$\dot{p}_1 = \dot{x}_2 = f_2(x_1,\ldots) = -\frac{\partial H}{\partial q_1}$$

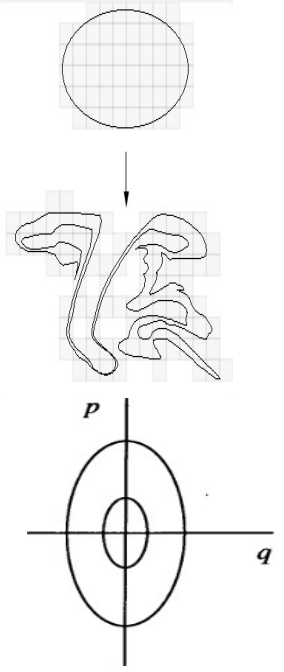
$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = \frac{\partial}{\partial q_1}\left(\frac{\partial H}{\partial p_1}\right) + \frac{\partial}{\partial p_1}\left(-\frac{\partial H}{\partial q_1}\right)$$

$$= \frac{\partial^2 H}{\partial q_1 \partial p_1} - \frac{\partial^2 H}{\partial p_1 \partial q_1} \qquad (8.3\text{-}5)$$

$$= 0 \longrightarrow \underbrace{div f} = 0 \longrightarrow \frac{1}{V}\frac{dV}{dt} = \sum_i \frac{\partial f_i}{\partial x_i} = 0$$

The final equality of Eq. (8.3-5) follows from the fact that the order of differentiation does not change the result for these second "cross" partial derivatives (unless H is an unphysically bizarre function of q and p). cfr.Teorema di Schwarz

This cancellation of terms continues pairwise for all the qs and ps. Hence, we conclude that for a Hamiltonian system, the volume occupied by a set of initial conditions does not change in time as the system evolves. The practical consequence of this unchanging volume is the fact that Hamiltonian systems do not have phase space attractors in the way dissipative systems do. As we shall see, this lack of attractors is both a simplification and a complication. Since we have no attractors, we do not need to worry about transients; that is, we do not need to let the trajectory run for some time so that it settles onto the appropriate attractor. This usually simplifies the process of finding the appropriate solution for the trajectories. On the other hand, we shall see that the lack of attractors means that trajectories starting with different initial conditions may behave quite differently as time goes on; there is no common attractor onto which they settle.

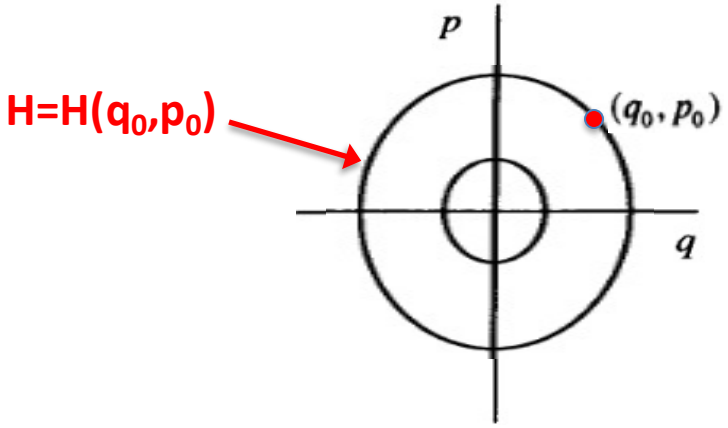


8.4 Constants of the Motion and Integrable Hamiltonians

In Section 8.2, we saw that the energy, represented by the Hamiltonian of a system, is conserved if the Hamiltonian does not depend on time explicitly. Let us flesh out some of the consequences of that result. If a trajectory in phase space starts at a point labeled (q_0, p_0), where q and p represent the entire set of $2N$ phase space coordinates, the system's energy is given by $H(q_0, p_0)$. As time goes on, the qs and ps evolve, but at any later time the energy will have the same value, namely

$$H(q(t), p(t)) = H(q_0, p_0) \tag{8.4-1}$$

where $(q(t), p(t))$ gives the phase-space trajectory originating from (q_0, p_0). Hence, value that "belongs to" that trajectory. Note that the converse statement is not necessarily true: there may be many different trajectories corresponding to the same energy value.



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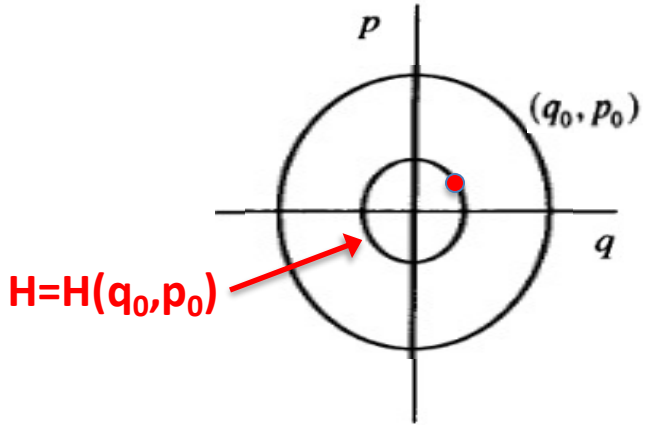

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In some Hamiltonian systems there are additional quantities whose values also remain constant as the trajectory evolves. Let us see why this is important by looking at a special case. Suppose one of the ps, say p_j does not change in time:

$$\dot{p}_j = 0 = -\frac{\partial H}{\partial q_j} \tag{8.4-2}$$

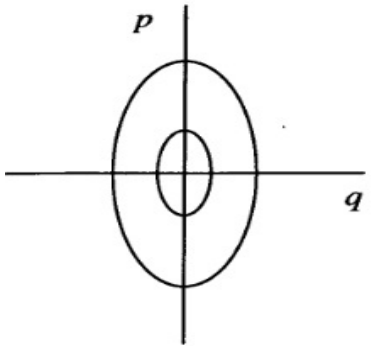
The only way the last term in Eq. (8.4-2) can be 0 for all $(q(t), p(t))$ values along the trajectory is to have $H(q, p)$ not depend on q_j at all! We have the general rule: The momentum p_j is a constant of the motion if, and only if, the Hamiltonian for a system does not depend on the corresponding q_j explicitly.

$$\dot{p}_j = 0 = -\frac{\partial H}{\partial q_j} \longrightarrow p_j = p_{j0} \quad \forall t$$ **Costante del Moto!**

In that case, a trajectory can be labeled by its value of $p_j = p_{j0}$ as well as by its energy value $H(q_0, p_0)$. When this occurs, the trajectories are limited not only to those regions of phase space associated with a particular energy value, they are also constrained by the value of p_j. Thus, the trajectories must "live" on a $2N$-k dimensional "surface" in phase space, where k is the number of conserved quantities.

For a special (and very limited, but theoretically important) class of Hamiltonian systems, there are as many constants of the motion as there are degrees of freedom. Such systems are called ***integrable***, for reasons that will shortly become obvious. However, in most cases, the constants of the motion are not the ps in terms of which we initially wrote the Hamiltonian. The constants of the motion, however, can always be expressed as functions of the original qs and ps. The constants of the motion are usually called the ***action variables*** and are commonly written as $J_i(q, p)$, $i = 1, 2, \ldots, N$. For an integrable Hamiltonian system, the phase space trajectories are confined to an N-dimensional surface in phase space.

Esempio di sistema integrabile:
N=1: 2D phase space e 1D energy surface



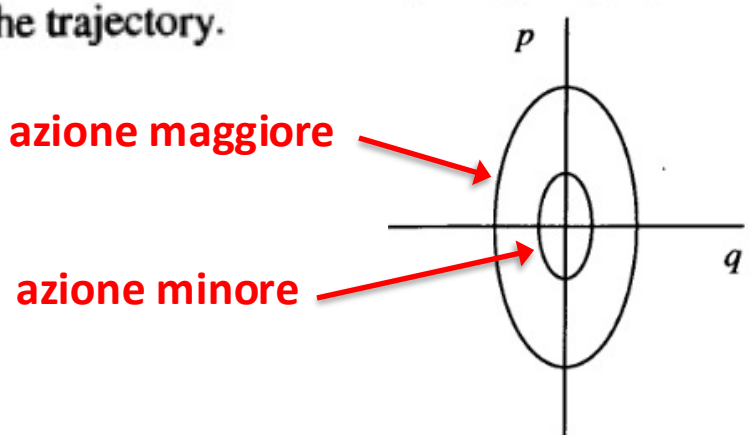
Associated with each $J_i(q, p)$ is another variable labeled $\theta_i(q, p)$. This new variable is called the corresponding ***angle variable***. (In an upcoming example, we shall see why these names are used.) The J_is and θ_is are chosen so that Hamilton's equations, expressed in terms of the J_is and θ_is have the same mathematical form as the original Hamilton's equations expressed in terms of the qs and ps:

Trasformazioni Canoniche

$$\begin{cases} \theta_i(t) = F(q_i, p_i) \\ J_i(t) = G(q_i, p_i) \end{cases} \longrightarrow \begin{cases} \dot{\theta}_i = \dfrac{\partial H(\theta, J)}{\partial J_i} \\ \dot{J}_i = -\dfrac{\partial H(\theta, J)}{\partial \theta_i} \end{cases} \qquad (8.4\text{-}3)$$

(Since the angle variable is dimensionless, we see that the action has units of energy multiplied by time, or equivalently, momentum multiplied by distance). If Eqs. (8.4-3) are satisfied, we say that the variables (θ, J) are related to the variables (q, p) by a ***canonical transformation***.

As we shall see, for a periodic trajectory in phase space, for which the trajectory forms a closed curve, the action has a nice geometric interpretation: The action associated with a periodic trajectory is proportional to the phase space area enclosed by the trajectory.



Caso Speciale: variabili azione-angolo per un sistema integrabile

$$\begin{cases} \dot{\theta}_i = \dfrac{\partial H(\theta, J)}{\partial J_i} \\ \dot{J}_i = -\dfrac{\partial H(\theta, J)}{\partial \theta_i} \end{cases}$$

The special case we are interested in is a canonical transformation that leads to a Hamiltonian that depends only on the J_is and not on the θ_i s. In that case, for all $i = 1, 2, \ldots, N$, we have

$$\dot{J}_i = 0 \tag{8.4-4}$$

and the J_is are the N constants of the motion.

A Hamiltonian system that satisfies Eqs. (8.4-3) and (8.4-4) is called (somewhat unfortunately) an ***integrable system***. The term *integrable* comes from the notion that the action J_i can be expressed as an integral over the motion of the system and that the corresponding equation for θ_i can be easily integrated.

Nota: The term *integrable* is a bit misleading because it seems to imply that the character of the system depends on our *ability* to find the appropriate canonical transformation or to do the required integral. In fact, one often finds phrases in the literature such as "A system is integrable, if we can find the canonical transformation . . ." In reality, the character of the system, that is the number of constants of the motion, is independent of our ability to find the appropriate canonical transformation.

Sistemi Integrabili

By expressing the desired canonical transformation in terms of a so-called ***Birkhoff Series*** and by examining the convergence properties of that series, one can determine (at least in principle) whether a given Hamiltonian system is integrable or nonintegrable (HEL80). If the system is ***nonintegrable***, it has fewer constants of the motion than degrees of freedom.

We will now list (without proof) some results, which tell us what kinds of Hamiltonian systems are integrable (HEL80).

1. All one-degree-of-freedom Hamiltonian systems, for which H is an infinitely differentiable (that is, "analytic") function of q and p, are integrable and the corresponding action J satisfies $H = \omega\ J$, where $\omega = \partial H / \partial J$.

2. All Hamiltonian systems for which Hamilton's equations are linear in q and p are integrable (via the so-called normal mode transformations).

3. All Hamiltonian systems with nonlinear Hamilton's equations that can be separated into uncoupled one-degree-of-freedom systems are integrable.

Let us now explore the consequences of having an integrable Hamiltonian, for which all the J_is are constants of the motion. In this case, the time dependence of θ_i is easy to find

$$(\dot{J}_i = 0) \qquad \dot{\theta}_i = \frac{\partial H}{\partial J_i} \equiv \omega_i(J) \tag{8.4-5}$$

The right-hand side of the previous equation defines what is called the angular frequency of the motion. For an integrable system, ω_i depends on the values of all the J_is, but because the J_is are independent of time, the ω_is are also independent of time. Thus, we can immediately write

$$\theta_i(t) = \omega_i t + \theta_i(0) \tag{8.4-6}$$

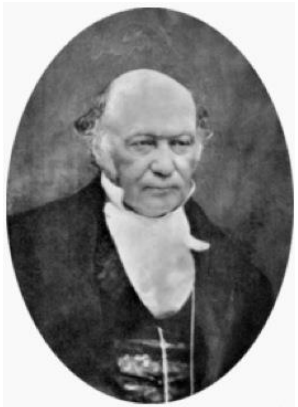
Hence, we see that if the system is integrable and if we can find the canonical transformations that give us Eqs. (8.4-3) and (8.4-4), then, amazingly, we have completely solved the dynamics of the system.

If we want to find the behavior of the system in terms of the original ps and qs, we can use the inverse of the canonical transformations to write

$$\begin{cases} q_i(t) = f(\theta, J) \\ p_i(t) = g(\theta, J) \end{cases} \tag{8.4-7}$$

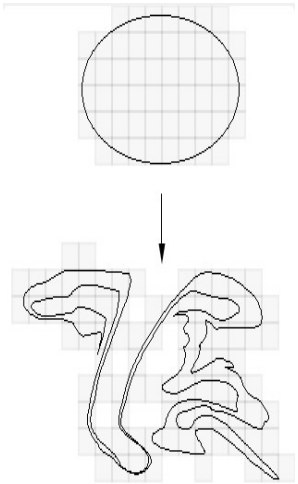
For a system that is bounded spatially, the qs and ps must be periodic functions of the θ_is since, according to Eq. (8.4-6), $\theta_i(t)$ increases without limit as $t \to \infty$.

Flussi Hamiltoniani ad un grado di libertà



William Rowan Hamilton (1805–1865)

$$\frac{1}{V}\frac{dV}{dt} = \sum_i \frac{\partial f_i}{\partial x_i} = 0$$



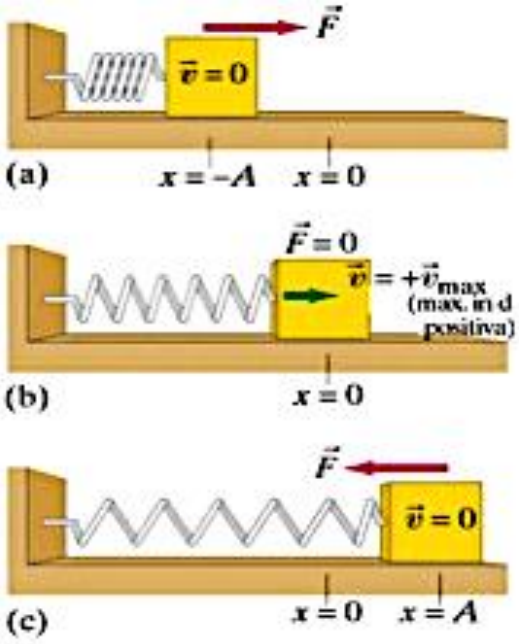
We will now study two examples of one-degree-of-freedom Hamiltonian systems and their phase space behavior.

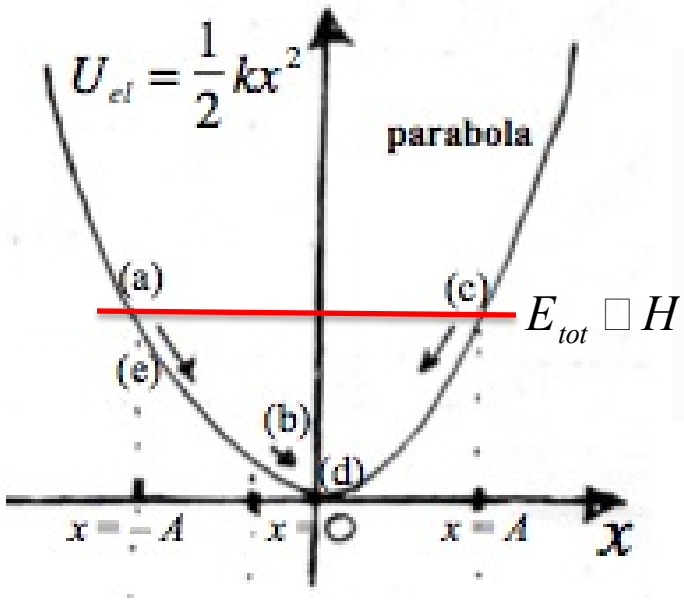
The Simple Harmonic Oscillator

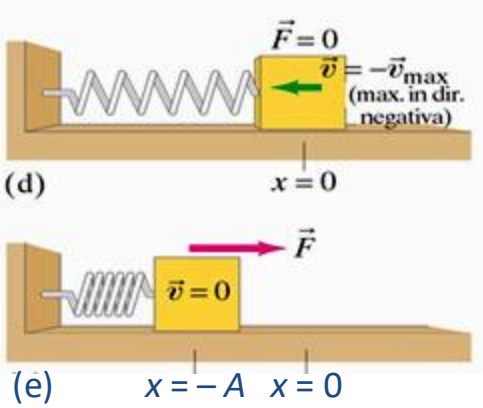
In Exercise 8.2-2, we learned that the Hamiltonian for a one-dimensional simple harmonic oscillator with mass m and spring constant k is

$$H(q,p)=\frac{p^2}{2m}+\tfrac{1}{2}kq^2 \qquad (8.4\text{-}8)$$

where q is the displacement of the oscillator from its equilibrium position. In this case, the numerical value of the Hamiltonian is the total mechanical energy of the system.







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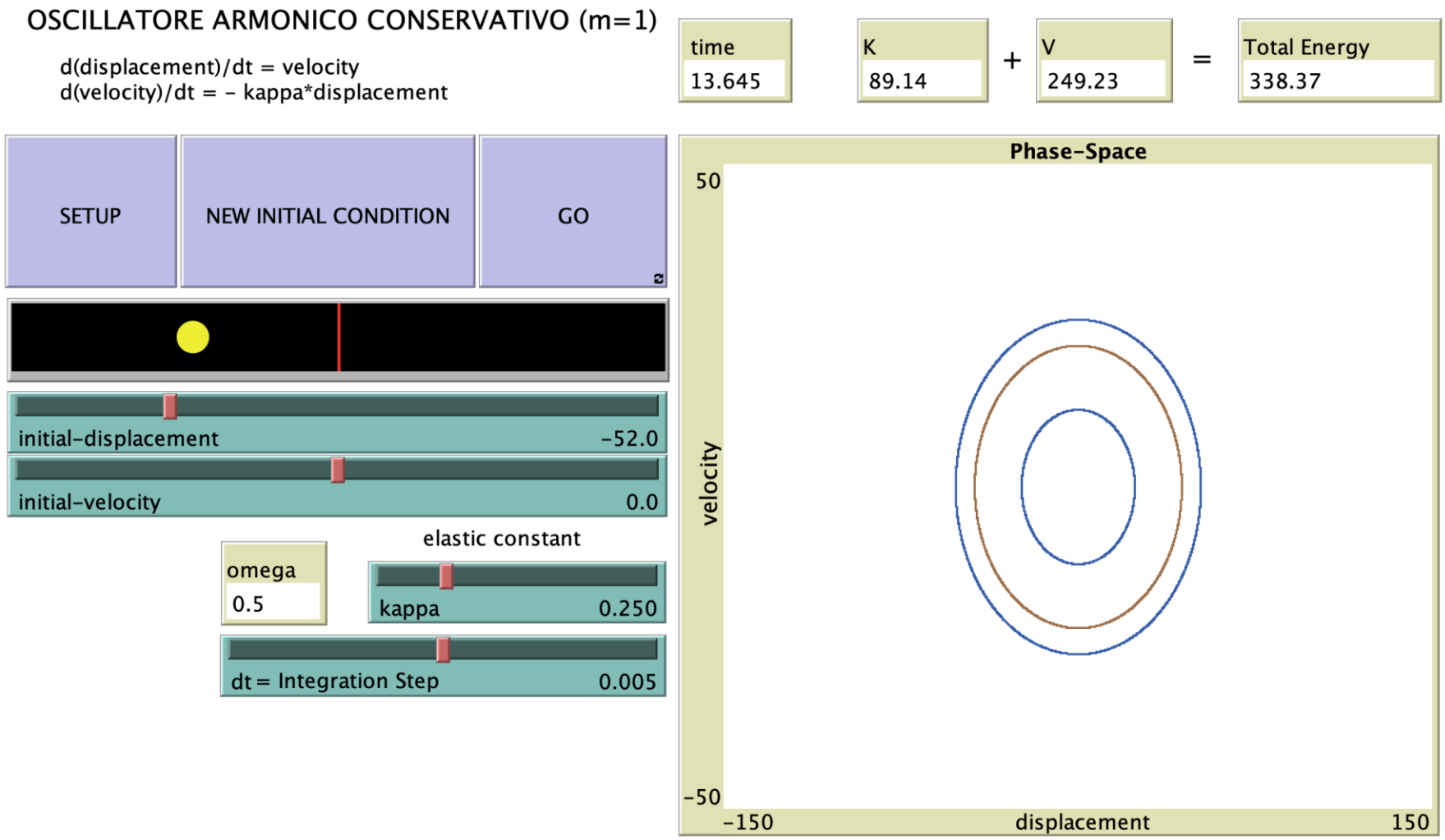
where q is the displacement of the oscillator from its equilibrium position. In this case, the numerical value of the Hamiltonian is the total mechanical energy of the system. The corresponding Hamilton's equations for the time evolution are

Hamilton's Equations

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} = \dfrac{p}{m} \\ \dot{p} = -\dfrac{\partial H}{\partial q} = -kq \end{cases} \tag{8.4-9}$$

The one (spatial) dimension simple harmonic oscillator model has one degree of freedom and its phase space is two-dimensional. Since the Hamiltonian is independent of time, the phase space trajectories must reside on a $2N-1 = 1$ dimensional "surface" (i.e., on a curve). The trajectories are closed curves because the motion is periodic. Each value of the energy is associated with a unique closed curve.

oscillatore-armonico.nlogo



Spazio delle Fasi 2D dell'Oscillatore Armonico

The phase space trajectories for the simple harmonic oscillator are ellipses with a larger ellipse associated with a larger value of the energy (Hamiltonian) of the system. If the phase space coordinates are suitably rescaled, as shown on the right in Fig. 8.1, then the trajectories become circles. As we shall see, the radius of each circle is equal to the square root of the value of the action associated with that trajectory. The corresponding angle variable gives the location of the trajectory point on the circle.

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -kq$$

$$\ddot{q}(t) = \frac{\dot{p}(t)}{m} = -\frac{k}{m} q(t)$$

$$q(t) = A\cos(\omega t + \theta_0)$$

$$p(t) = m\dot{q}(t) = -m\omega A\sin(\omega t + \theta_0)$$

$$\omega = \sqrt{\frac{k}{m}}$$

Legge di Hooke

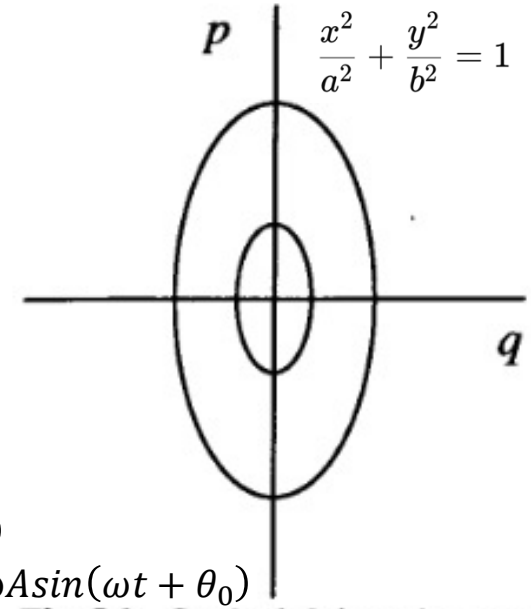


Fig. 8.1. On the left is a phase space trajectory for the simple harmonic oscillator. Each ellipse is associated with a particular value of the energy. A larger ellipse has a larger value of the energy. By rescaling the variables, the trajectories become circles whose radii are equal to $\sqrt{J}$, the square root of the action value associated with that trajectory. The corresponding angle variable θ locates the point on the trajectory.

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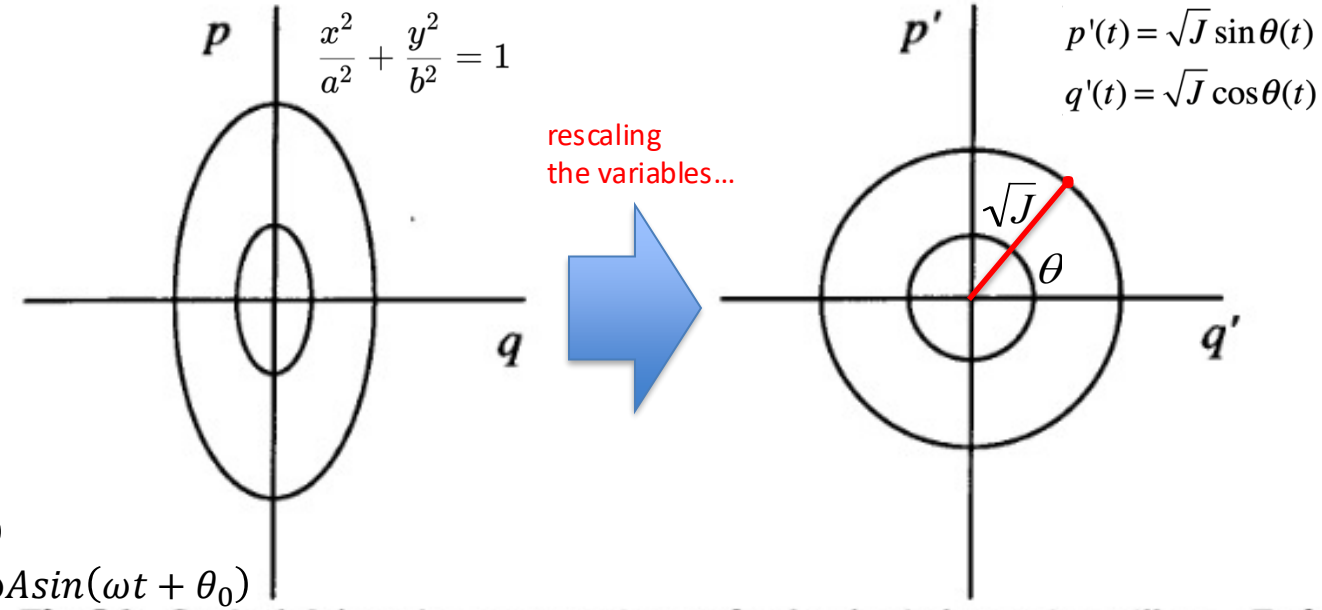

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For the simple harmonic oscillator, we know that the angular frequency of the oscillatory motion is given by $\omega = \sqrt{k/m}$. Since this is a one-degree-of-freedom system or since Hamilton's equations are linear, we expect that this system is integrable. The one constant of the motion is the Hamiltonian (energy) or some multiple thereof. Hence, we can write the action J as

$$\dot{\theta}_i = \frac{\partial H}{\partial J_i} \equiv \omega_i(J) \qquad \dot{J}_i = 0$$

$$H = \omega J \longrightarrow J = \frac{H}{\omega} = \frac{p^2}{2m\omega} + \frac{kq^2}{2\omega} \qquad (8.4\text{-}10)$$

$$\omega = \sqrt{\frac{k}{m}}$$

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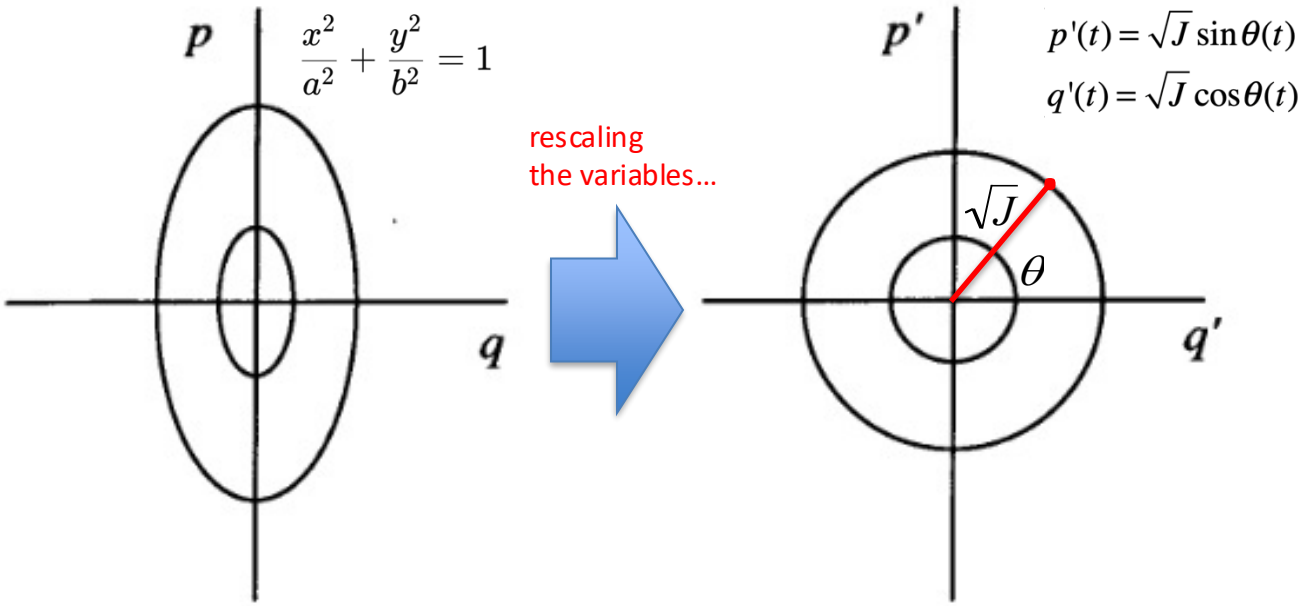

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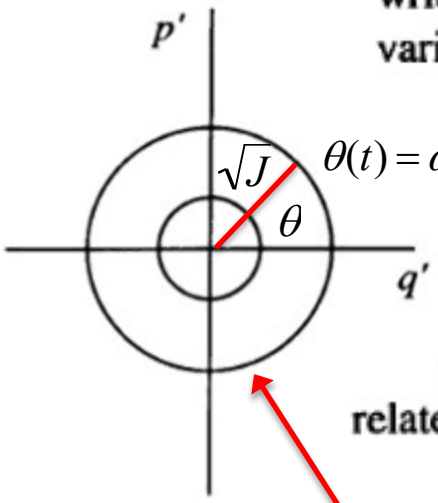
$$H = \omega J \longrightarrow J = \frac{H}{\omega} = \frac{p^2}{2m\omega} + \frac{kq^2}{2\omega} \tag{8.4-10}$$

If we use $p/\sqrt{2m\omega}$ and $q\sqrt{m\omega/2}$ as the phase space variables, then the trajectories will be circles with radii equal to $\sqrt{J}$. To complete the story, we can write the original phase space variables p and q in terms of the action-angle variables (with θ positive going counterclockwise from the positive q axis):



$$\begin{cases} p(t) = \sqrt{2m\omega J}\sin\theta(t) \\ q(t) = \sqrt{2J/(m\omega)}\cos\theta(t) \end{cases} \rightarrow \begin{cases} p'(t) = \sqrt{J}\sin\theta(t) \\ q'(t) = \sqrt{J}\cos\theta(t) \end{cases} \tag{8.4-11}$$

As Exercise 8.4-1 shows, the action associated with a closed trajectory is related to the phase space area enclosed by the trajectory. In general, we may write

$$J = \frac{1}{2\pi}\oint p\, dq \tag{8.4-12}$$

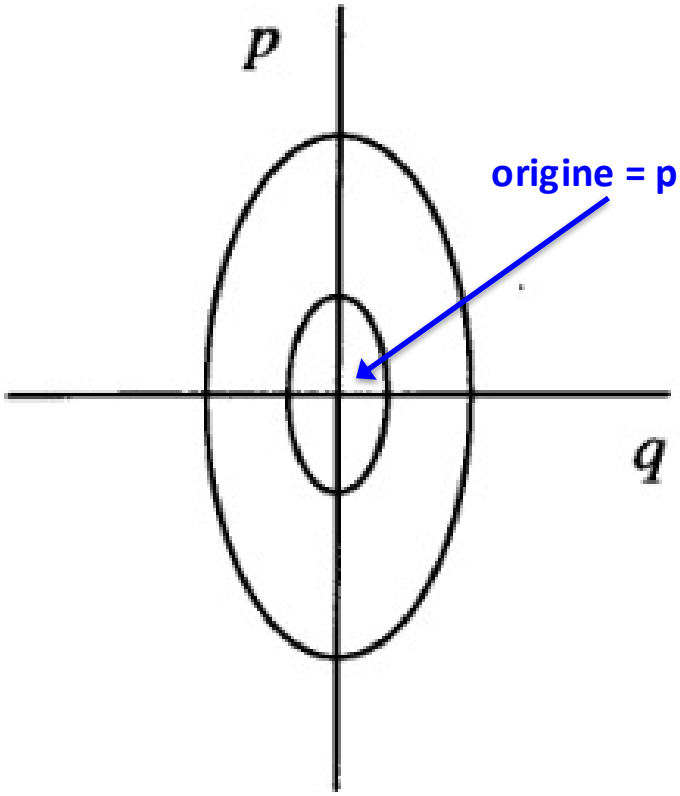
where the symbol $\oint$ means that the integral is taken around the closed path of the trajectory.

Studio dei punti fissi dell'Oscillatore Armonico non smorzato e smorzato

NOTA: Si noti che la struttura delle equazioni di Hamilton ci permette di applicare anche ai sistemi Hamiltoniani la procedura introdotta nei sistemi dissipativi per trovare e studiare i punti fissi, con la differenza che qui i punti fissi non saranno più attrattori della dinamica.

Note that the simple harmonic oscillator model has only one fixed point, namely $(p = 0, q = 0)$. In the language of Hamiltonian dynamics this kind of fixed point is called an ***elliptic point*** because the trajectories near the fixed point are ellipses.

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} = \dfrac{p}{m} = 0 \\ \dot{p} = -\dfrac{\partial H}{\partial q} = -kq = 0 \end{cases}$$



Oscillatore Armonico Conservativo

$$\begin{cases} \dot{q} = p \quad (m=1) \\ \dot{p} = -kq \end{cases}$$

$$\ddot{q} + kq = 0$$

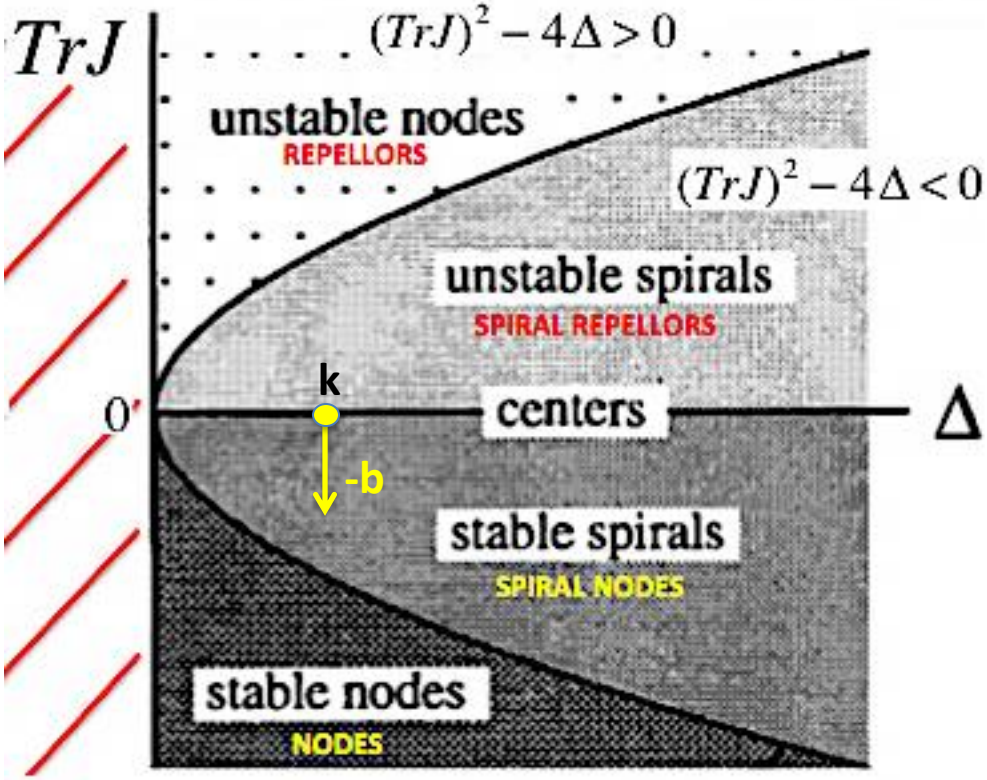
Oscillatore Armonico Smorzato

$$\begin{cases} \dot{q} = p \quad (m=1) \\ \dot{p} = -bp - kq \end{cases}$$

$$\ddot{q} + b\dot{q} + kq = 0$$

Punto Ellittico ➡ **Attrattore a punto fisso**

$q = 0\ ,\ p = 0$ $\qquad$ $q = 0\ ,\ p = 0$



WolframAlpha

{{0,1},{-k,-b}}

Input:

$$\begin{pmatrix} 0 & 1 \\ -k & -b \end{pmatrix}$$

$$J = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

Dimensions:

2 (rows) × 2 (columns)

Determinant:

k

Trace:

$-b$

Parametro di smorzamento

$$\begin{cases} b = 0 & \text{OA conservativo} \\ b > 0 & \text{OA smorzato} \\ b \gg 0 \end{cases}$$

Characteristic polynomial:

$b\,x + k + x^2$

Eigenvalues:

$$\lambda_1 = \frac{1}{2}\left(-\sqrt{b^2 - 4k} - b\right)$$

$$\lambda_2 = \frac{1}{2}\left(\sqrt{b^2 - 4k} - b\right)$$

Oscillatore Armonico Conservativo

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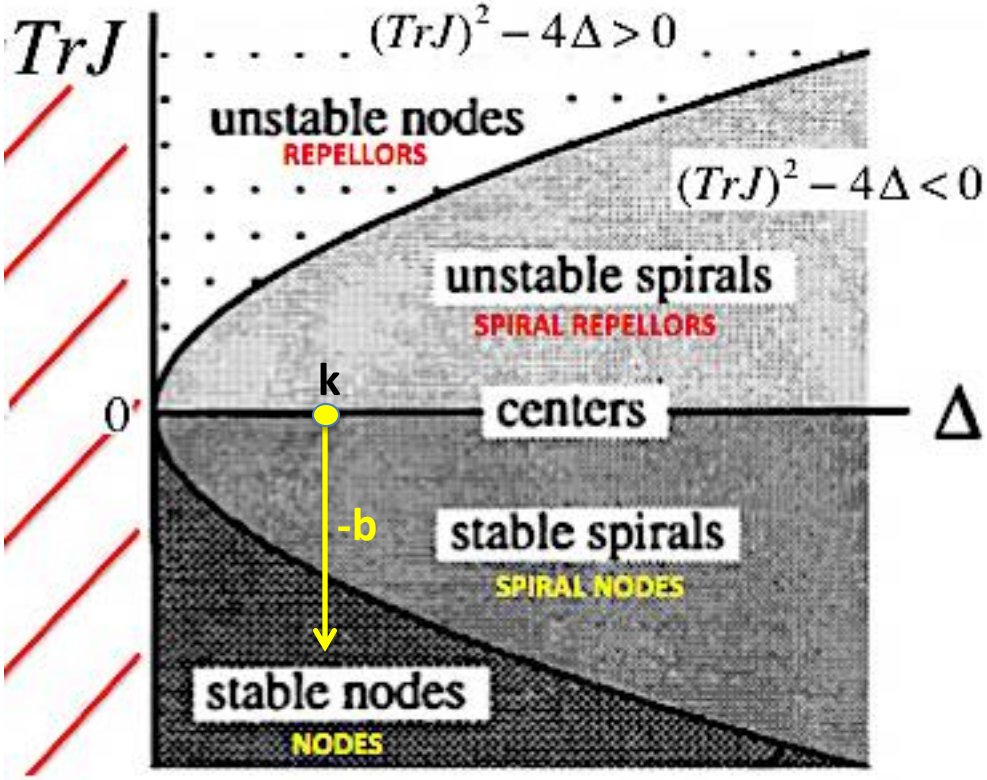

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2 (rows) × 2 (columns)

Determinant:

k

Trace:

$-b$

Parametro di smorzamento

$$\begin{cases} b = 0 & \text{OA conservativo} \\ b > 0 & \text{OA smorzato} \\ b \gg 0 & \text{OA sovrasmorzato} \end{cases}$$

Characteristic polynomial:

$b\,x + k + x^2$

Eigenvalues:

$$\lambda_1 = \frac{1}{2}\left(-\sqrt{b^2 - 4k} - b\right)$$

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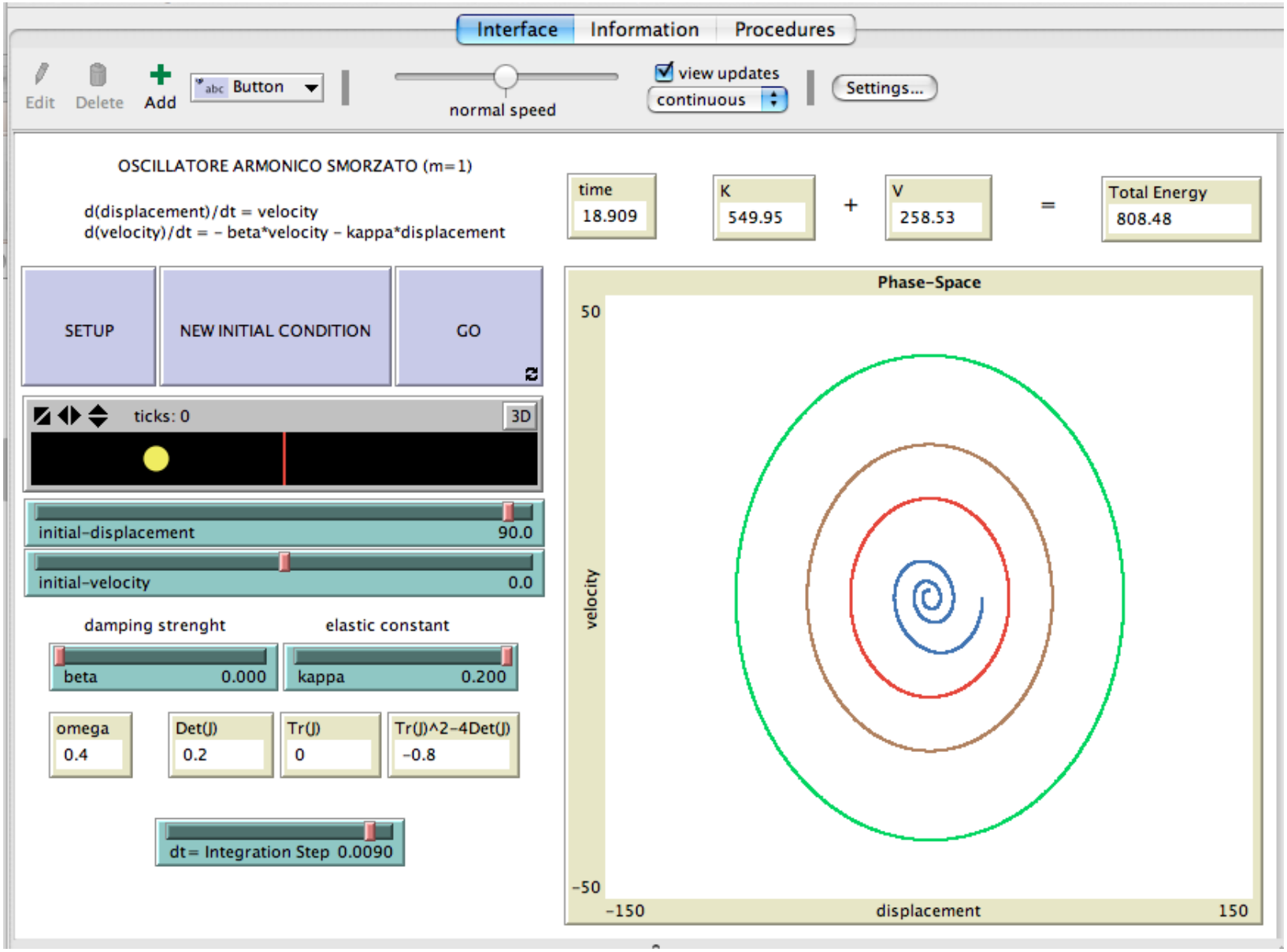
oscillatore-armonico-smorzato



CIRCUITO_RLC_SERIE.nlogo

CIRCUITO RLC IN SERIE (L=1)

dQ/dt = I (corrente)
dI/dt = - R*I - (1/C)*Q

tempo	(1/2)LI^2	+	Q^2/2C	=	Energia Totale
10.69	59.02		498.79		558

SETUP | NUOVA CONDIZ. INIZIALE | START

carica-iniziale 60

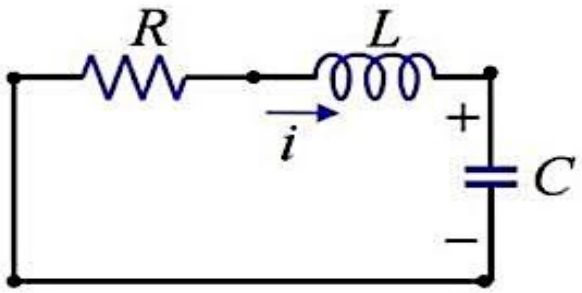
corrente-iniziale 0.0

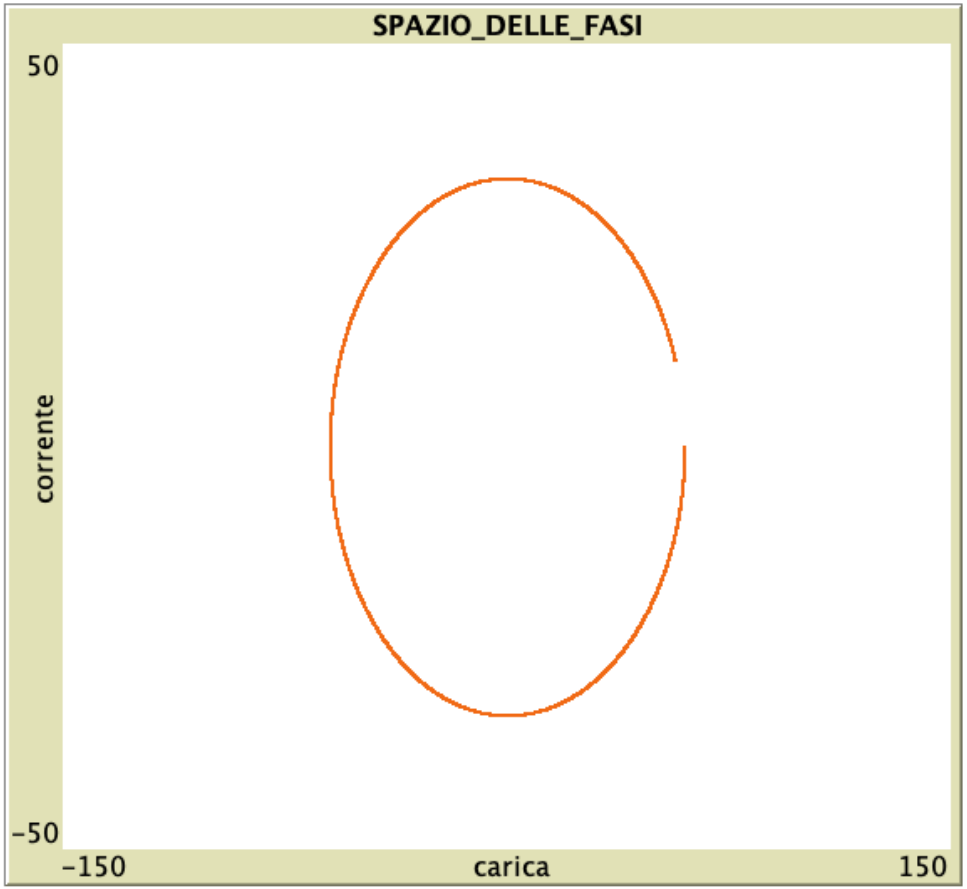
Resistenza 0.00

capacità-inversa 0.31

Resistenza-critica	pulsazione
1.11	0.56

dt = passo di integrazione 0.0020





Ex.1 ROMEO E GIULIETTA

Il libro di Strogatz suggerisce di studiare, come esercizio, un **sistema dinamico lineare** a due dimensioni che descrive, al variare dei parametri, la **variazione temporale dell'amore o dell'odio tra due partner coinvolti in una relazione romantica**.

Definiamo x(t) come l'amore (o l'odio nel caso in cui sia negativo) di Romeo nei confronti di Giulietta al tempo "t" e y(t) l'amore (o l'odio) di Giulietta nei confronti di Romeo. Così abbiamo le seguenti **due equazioni differenziali del primo ordine:**

Sistema Lineare

Romeo $\dot{x} = ax + by$

Giulietta $\dot{y} = cx + dy$

$$\begin{cases} ax + by = 0 \\ cx + dy = 0 \end{cases}$$

Quando il determinante della matrice dei coefficienti è diverso da zero, il sistema ha esattamente un **unico punto fisso**, che può essere trovato risolvendo il sistema di equazioni lineari. In questo caso il punto fisso si trova **all'origine** (0,0) poichè non ci sono termini costanti nelle equazioni (sistema omogeneo).

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Lo Jacobiano coincide con la matrice dei coefficienti e quindi descrive il comportamento del sistema anche a **distanze maggiori** dal punto fisso (non richiede alcuna espansione in serie di Taylor).

Nei sistemi dinamici lineari non possono emergere cicli limite. I cicli limite sono una caratteristica dei sistemi dinamici non lineari. Un sistema lineare può avere un comportamento periodico, come nel caso di un oscillatore armonico ideale senza attrito (quindi conservativo), ma questo comportamento non rappresenta un ciclo limite bensì un'orbita ad energia costante: in questo caso la traccia dello Jacobiano sarà zero e il punto fisso sarà un punto ellittico. Se invece la traccia è maggiore di zero, il sistema torna ad essere dissipativo e il punto fisso può essere un nodo standard o un nodo a spirale a seconda del segno del radicando nell'equazione caratteristica.