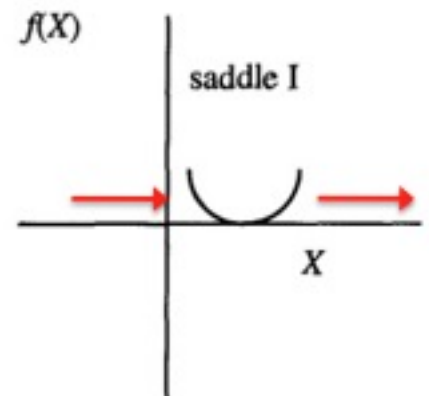
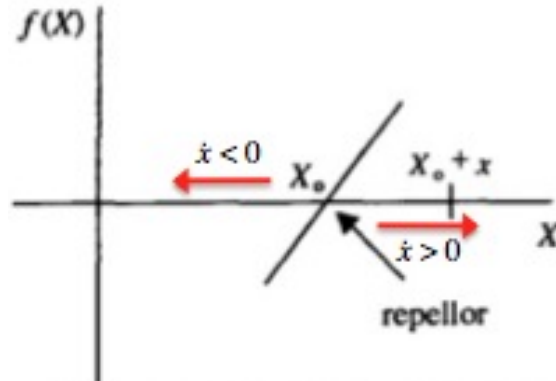
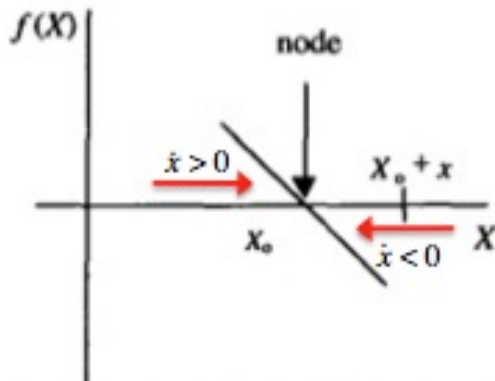


Flussi dissipativi in una dimensione

$$\frac{1}{L} \frac{dL}{dt} = \frac{1}{L} [f(X_B) - f(X_A)] = \frac{df(X)}{dX} < 0$$

fixed points (dim.0)



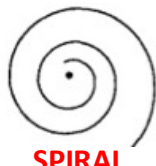
Flussi dissipativi in due dimensioni

$$\frac{1}{A} \frac{dA}{dt} = \frac{\partial f_1}{\partial X_1} + \frac{\partial f_2}{\partial X_2} < 0$$

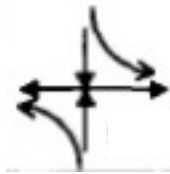
fixed points (dim.0)



NODE



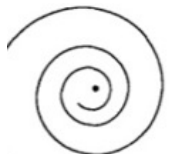
SPIRAL
NODE



SADDLE
POINTS

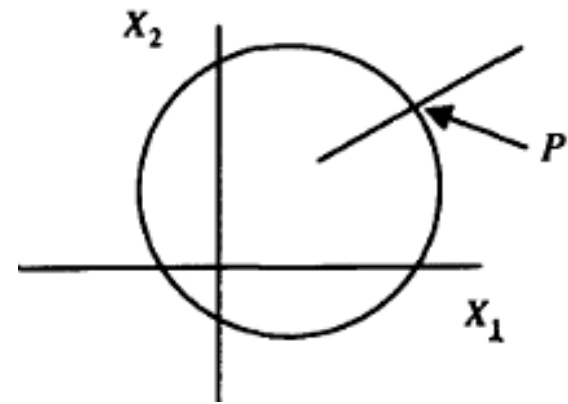


REPELLOR



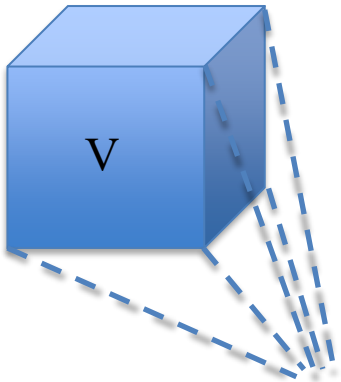
SPIRAL
REPELLOR

limit cycles (dim.1)



Flussi dissipativi in tre dimensioni

Cluster di
condizioni iniziali



ATTRATTORI

$$\frac{1}{V} \frac{dV}{dt} = \sum_{i=1}^N \frac{\partial f_i}{\partial x_i} \equiv \text{div}(f) < 0$$

fixed points (dim.0)

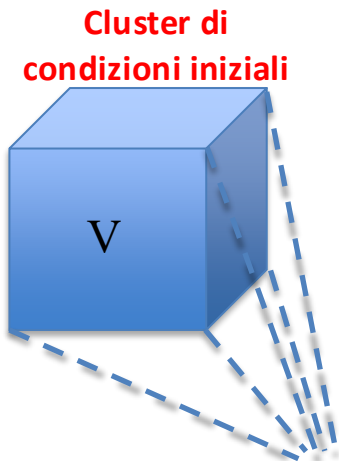
limit cycles (dim.1)

quasiperiodic attractors (dim.2)

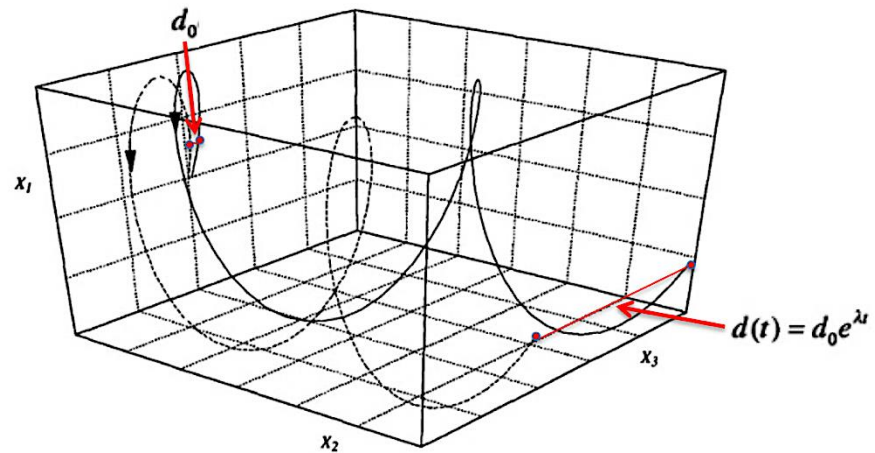
chaotic attractors (fractal dimension between 2 and 3)

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, x_3) \\ \dot{x}_2 = f_2(x_1, x_2, x_3) \\ \dot{x}_3 = f_3(x_1, x_2, x_3) \end{cases}$$

Flussi dissipativi in tre dimensioni



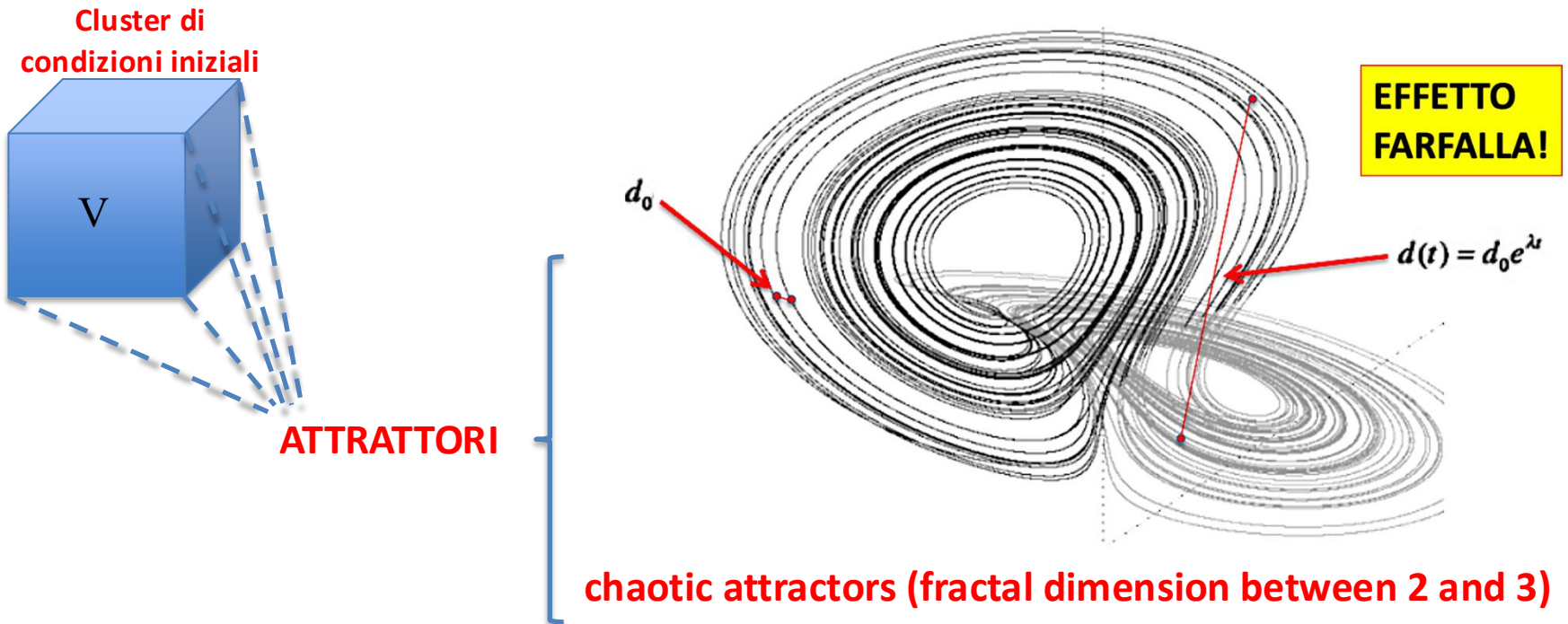
ATTRATTORI



1. no intersection of different trajectories;
2. bounded trajectories;
3. exponential divergence of nearby trajectories.

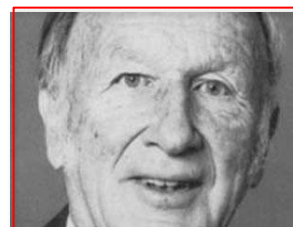
chaotic attractors (fractal dimension between 2 and 3)

Flussi dissipativi in tre dimensioni



4.4 Three-Dimensional Dynamical Systems

We will now introduce some of the formalism for the description of a dynamical system with three state variables. We call a dynamical system three-dimensional if it has three independent dynamical variables, the values of which at a given instant of time uniquely specify the state of the system. We assume that we can write the time-evolution equations for the system in the form of the standard set of first-order ordinary differential equations. (Dynamical systems modeled by iterated map functions will be discussed in Chapter 5.) Here we will use x with a subscript 1, 2, or 3 to identify the variables. This formalism can then easily be generalized to any number of dimensions simply by increasing the numerical range of the subscripts. The differential equations take the form



$$\begin{aligned}\dot{X} &= p(Y - X) \\ \dot{Y} &= -XZ + rX - Y \\ \dot{Z} &= XY - bZ\end{aligned}$$

$$\left\{ \begin{aligned}\dot{x}_1 &= f_1(x_1, x_2, x_3) \\ \dot{x}_2 &= f_2(x_1, x_2, x_3) \\ \dot{x}_3 &= f_3(x_1, x_2, x_3)\end{aligned} \right. \quad (4.4-1)$$

The Lorenz model equations of Chapter 1 are of this form. Note that the three functions f_1 , f_2 , and f_3 do not involve time explicitly; again, we say that the system is autonomous.

As an aside, we note that some authors like to use a symbolic “vector” form to write the system of equations:

$$\vec{\dot{x}} = \vec{f}(\vec{x}) \quad (4.4-2)$$

Here \vec{x} stands for the three symbols x_1, x_2, x_3 , and \vec{f} stands for the three functions on the right-hand side of Eqs. (4.4-1).

The differential equations describing two-dimensional systems subject to a time-dependent “force” (and hence nonautonomous) can also be written in the form of Eq. (4.4-1) by making use of the “trick” introduced in Chapter 3: Suppose that the two-dimensional system is described by equations of the form

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2, t) \\ \dot{x}_2 &= f_2(x_1, x_2, t)\end{aligned}\tag{4.4-3}$$

The trick is to introduce a third variable, $x_3 = t$. The three “autonomous” equations then become

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2, x_3) \\ \dot{x}_2 &= f_2(x_1, x_2, x_3) \\ \dot{x}_3 &= 1\end{aligned}\tag{4.4-4}$$

which are of the same form as Eq. (4.4-1). As we shall see, this trick is particularly useful when the time-dependent term is periodic in time.

Exercise 4.4-1. The “forced” van der Pol equation is used to describe an electronic triode tube circuit subject to a periodic electrical signal. The equation for $q(t)$, the charge oscillating in the circuit, can be put in the form

$$\frac{d^2 q}{dt^2} + \gamma(q) \frac{dq}{dt} + q(t) = g \sin \omega t$$

Use the trick introduced earlier to write this equation in the standard form of Eq. (4.4-1).

4.5 Fixed Points in Three Dimensions (dim = 0)

The fixed points of the system of Eqs. (4.4-1) are found, of course, by setting the three time derivatives equal to 0. [Two-dimensional forced systems, even if written in the three-dimensional form (4.4-4), do not have any fixed points because, as the last of Eqs. (4.4-4) shows, we never have $\dot{x}_3 = t = 0$. Thus, we will need other techniques to deal with them.] The nature of each of the fixed points is determined by the three characteristic values of the Jacobian matrix of partial derivatives evaluated at the fixed point in question. The Jacobian matrix is

Trovare i punti fissi:

$$\left\{ \begin{array}{l} \dot{x}_1 = f_1(x_1, x_2, x_3) = 0 \\ \dot{x}_2 = f_2(x_1, x_2, x_3) = 0 \\ \dot{x}_3 = f_3(x_1, x_2, x_3) = 0 \end{array} \right. \quad \begin{array}{l} \text{Per ogni punto} \\ \text{fisso calcolare:} \end{array} \quad \rightarrow \quad J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{pmatrix} \quad \begin{array}{l} \text{Che in generale} \\ \text{dipender\`a anche} \\ \text{dal valore degli} \\ \text{eventuali} \\ \text{parametri di} \\ \text{controllo...} \end{array} \quad (4.5-1)$$

In finding the characteristic values of this matrix, we will generally have a cubic equation, whose roots will be the three characteristic values labeled $\lambda_1, \lambda_2, \lambda_3$.

Some mathematical details: The standard theory of cubic equations tells us that a cubic equation of the form

$$\lambda^3 + p\lambda^2 + q\lambda + r = 0 \quad (4.5-2)$$

can be changed to the “standard” form


$$x^3 + ax + b = 0 \quad (4.5-3)$$

by the use of the substitutions

$$x = \lambda + p/3$$
$$a = \frac{1}{3}(3q - p^2) \quad (4.5-4)$$

$$b = \frac{1}{27}(2p^3 - 9qp + 27r)$$

If we now introduce

$$s = \left(\frac{b^2}{4} + \frac{a^3}{27} \right)$$

$$A = (-b/2 + \sqrt{s})^{\frac{1}{3}} \quad (4.5-5)$$
$$B = (-b/2 - \sqrt{s})^{\frac{1}{3}}$$

the three roots of the x equation can be written as

$$\begin{aligned}\lambda_1 &= A + B \\ \lambda_2 &= -\left(\frac{A+B}{2}\right) + \left(\frac{A-B}{2}\right)\sqrt{-3} \\ \lambda_3 &= -\left(\frac{A+B}{2}\right) - \left(\frac{A-B}{2}\right)\sqrt{-3}\end{aligned}\tag{4.5-6}$$

from which the characteristic values for the matrix can be found by working back through the set of substitutions. Most readers will be greatly relieved to know that we will not make explicit use of these equations. But it is important to know the form of the solutions.



Enter what you want to calculate or know about:

You will immediately find both eigenvalues and eigenvectors of $J...$



Examples Random

the three roots of the x equation can be written as

$$\begin{aligned}\lambda_1 &= A + B \\ \lambda_2 &= -\left(\frac{A+B}{2}\right) + \left(\frac{A-B}{2}\right)\sqrt{-3} \\ \lambda_3 &= -\left(\frac{A+B}{2}\right) - \left(\frac{A-B}{2}\right)\sqrt{-3}\end{aligned}\tag{4.5-6}$$

from which the characteristic values for the matrix can be found by working back through the set of substitutions. Most readers will be greatly relieved to know that we will not make explicit use of these equations. But it is important to know the form of the solutions.

There are three cases to consider:

- “standard” form
 $x^3 + ax + b = 0$
 $s = \left(\frac{b^2}{4} + \frac{a^3}{27}\right)$
1. The three characteristic values are real and unequal ($s < 0$).
 2. The three characteristic values are real and at least two are equal ($s = 0$).
 3. There is one real characteristic value and two complex conjugate values ($s > 0$).

Case 2 is just a borderline case and need not be treated separately.

Punti Fissi in uno Spazio degli Stati a Tre Dimensioni

The four basic types of fixed points for a three-dimensional state space are:

1. **Node.** All the characteristic values are real and negative. All trajectories in the neighborhood of the node are attracted toward the fixed point without looping around the fixed point.
 - 1s. **Spiral Node.** All the characteristic values have negative real parts but two of them have nonzero imaginary parts (and in fact form a complex conjugate pair). The trajectories spiral around the node on a “surface” as they approach the node.

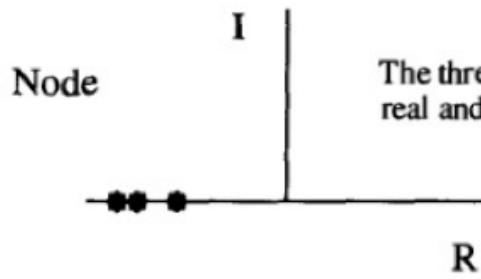
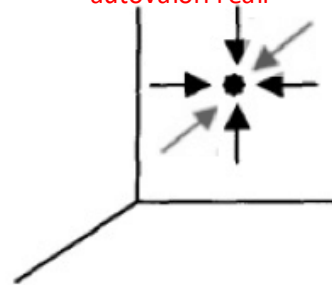
Equazione caratteristica:

$$\lambda^3 + p\lambda^2 + q\lambda + r = 0$$

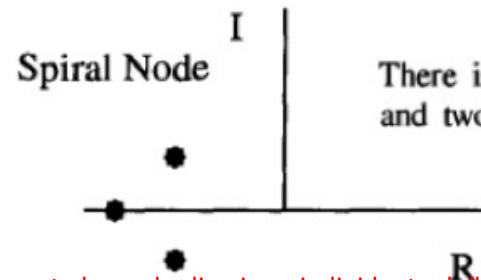
“standard” form
 $x^2 + ax + b = 0$

$$s = \left(\frac{b^2}{4} + \frac{a^3}{27} \right)$$

L'attrazione avviene linearmente lungo le tre direzioni individuate dai tre autovettori associati agli autovalori reali



The three characteristic values are real and unequal ($s < 0$).



There is one real characteristic value and two complex conjugate values ($s > 0$).

L'attrazione avviene linearmente lungo la direzione individuata dall'autovettore associato all'autovalore reale, ma anche spiraleggiando sul piano individuato dai due autovettori associati agli autovalori complessi

Punti Fissi in uno Spazio degli Stati a Tre Dimensioni

The four basic types of fixed points for a three-dimensional state space are:

2. **Repellor.** All the characteristic values are real and positive. All trajectories in the neighborhood of the repellor diverge from the repellor.
- 2s. **Spiral Repellor.** All the characteristic values have positive real parts, but two of them have nonzero imaginary parts (and in fact form a complex conjugate pair). Trajectories spiral around the repellor (on a “surface”) as they are repelled from the fixed point.

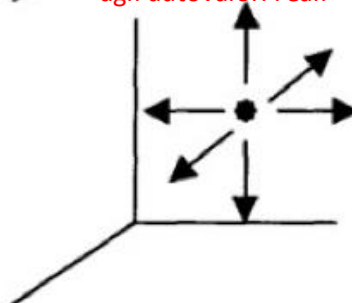
La repulsione avviene linearmente lungo le tre direzioni individuate dai tre autovettori associati agli autovalori reali

Equazione caratteristica:

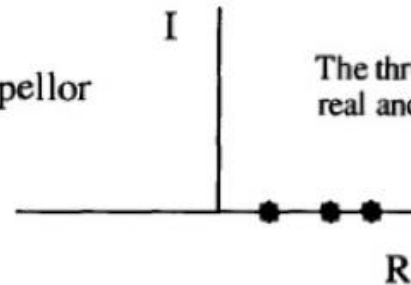
$$\lambda^3 + p\lambda^2 + q\lambda + r = 0$$

“standard” form
 $x^2 + ax + b = 0$

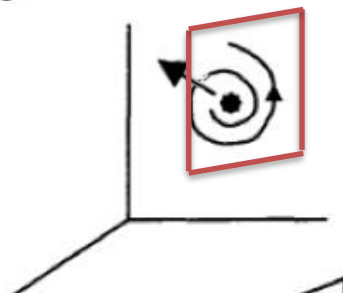
$$s = \left(\frac{b^2}{4} + \frac{a^3}{27} \right)$$



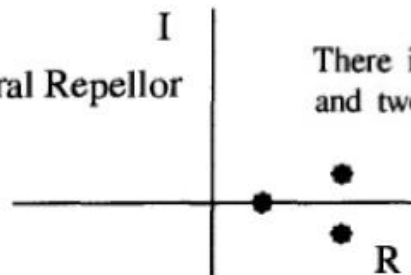
Repellor



The three characteristic values are real and unequal ($s < 0$).



Spiral Repellor



There is one real characteristic value and two complex conjugate values ($s > 0$).

La repulsione avviene linearmente lungo la direzione individuata dall'autovettore associato all'autovalore reale, ma anche spiraleggiando sul piano individuato dai due autovettori associati agli autovalori complessi

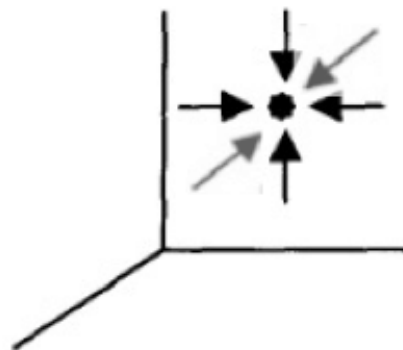
Punti Fissi in uno Spazio degli Stati a Tre Dimensioni

For state spaces with three or more dimensions, it is common to specify the so-called *index* of a fixed point.

The *index* of a fixed point is defined to be the number of characteristic values of that fixed point whose real parts are positive.

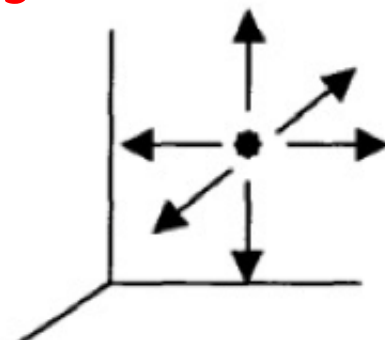
In more geometric terms, the index is equal to the spatial dimension of the out-set of that fixed point. For a node (which does not have an out-set), the index is equal to 0. For a repellor, the index is equal to 3 for a three-dimensional state space. A saddle point can have either an index of 1, if the out-set is a curve, or an index of 2, if the out-set is a surface as shown in Fig. 4.3.

Index = 0



Node

Index = 3



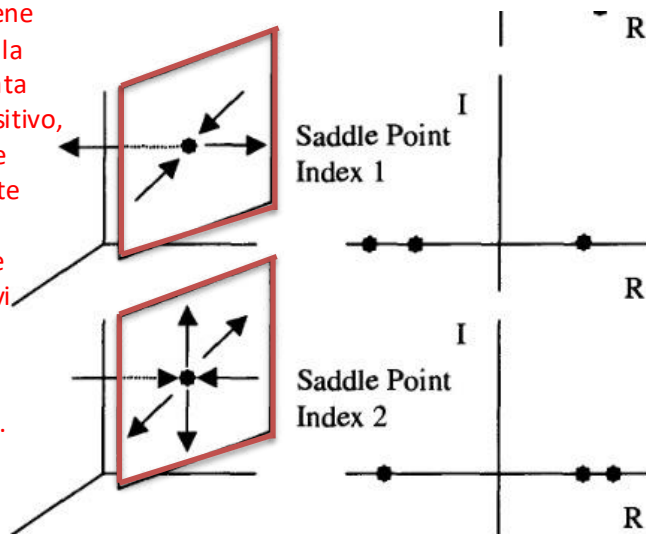
Repellor

Punti Fissi in uno Spazio degli Stati a Tre Dimensioni

3. **Saddle point — index-1.** All characteristic values are real. One is positive and two are negative. Trajectories approach the saddle point on a surface (the in-set) and diverge along a curve (the out-set).
- 3s. **Spiral Saddle Point — index-1.** The two characteristic values with negative real parts form a complex conjugate pair. Trajectories spiral around the saddle point as they approach on the in-set surface.
4. **Saddle point — index-2.** All characteristic values are real. Two are positive and one is negative. Trajectories approach the saddle point on a curve (the in-set) and diverge from the saddle point on a surface (the out-set).
- 4s. **Spiral Saddle Point — index-2.** The two characteristic values with positive real parts form a complex conjugate pair. Trajectories spiral around the saddle point on a surface (the out-set) as they diverge from the saddle point.

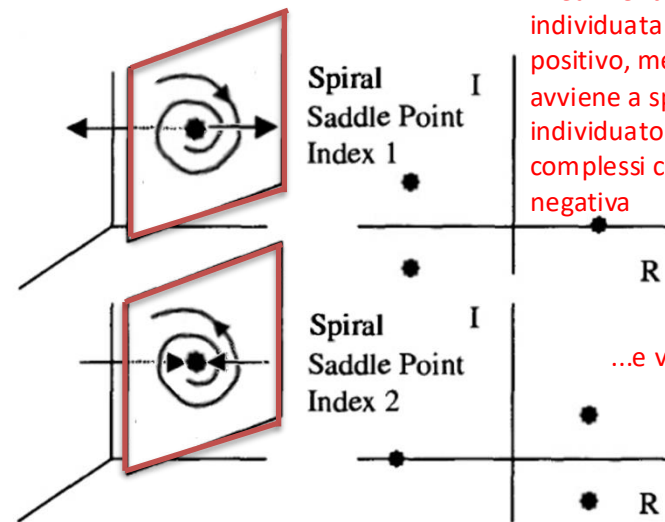
La repulsione avviene linearmente lungo la direzione individuata dall'autovalore positivo, mentre l'attrazione avviene linearmente lungo il piano individuato dai due autovettori negativi

...e viceversa...



La repulsione avviene linearmente lungo la direzione individuata dall'autovalore positivo, mentre l'attrazione avviene a spirale lungo il piano individuato dai due autovettori complessi con parte reale negativa

...e viceversa...



4.6 Limit Cycles and Poincaré Sections ($\text{dim} = 1$)

As we saw in Chapter 3, dynamical systems in two (and higher) dimensions can also settle into long-term behavior associated with repetitive, periodic limit cycles. We also learned that the Poincaré section technique can be used to reduce the dimensionality of the description of these limit cycles and to make their analysis simpler.

First, we focus on the construction of a Poincaré section for the system. For a three-dimensional state space, the Poincaré section is generated by choosing a Poincaré plane (a two-dimensional surface) and recording on that surface the points at which a given trajectory cuts through that surface. (In most cases the choice of plane is not crucial as long as the trajectories cut the surface *transversely*, that is, the trajectories do not run parallel or almost parallel to the surface as they pass through; see Fig. 4.4.) For autonomous systems, such as the Lorenz model equations, we choose some convenient plane in the state space, say, the XY plane for the Lorenz equations. When a trajectory crosses that plane passing from, for example, negative Z values to positive Z values, we record that crossing point.

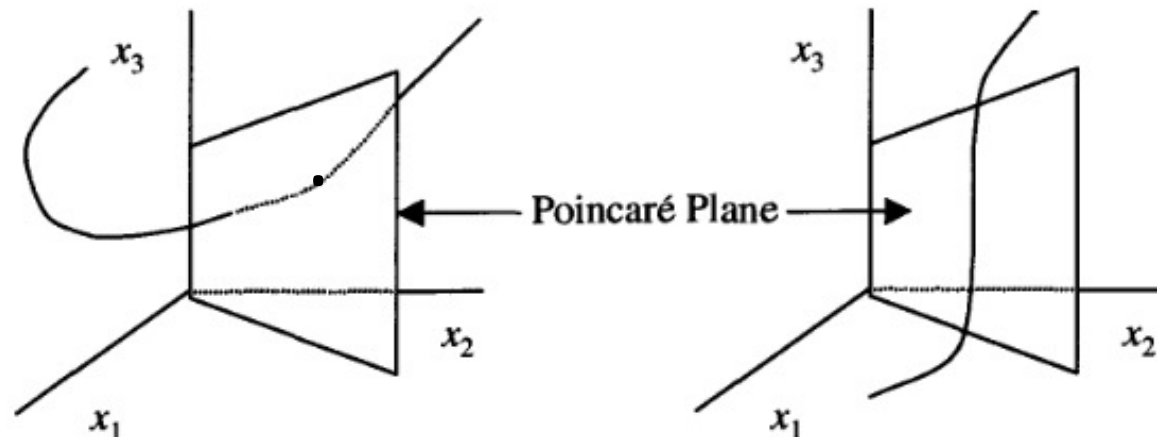


Fig. 4.4. A Poincaré section for a three-dimensional state space. On the left the trajectory crosses the Poincaré plane transversely. On the right the intersection is not transverse because the trajectory runs parallel to the plane for some distance.

In later discussions, it will be useful to indicate on the Poincaré section the record of trajectory intersections with the plane as trajectories approach or diverge from the periodic points. For example, Fig. 4.6 shows a sequence of points P_0, P_1, P_2, \dots as a trajectory approaches an attracting limit cycle in a three-dimensional state space. (Compare Fig. 4.6 with Fig. 3.13.) The reader should be warned that in some diagrams found in the literature this series of dots will be connected with a smooth curve intersecting (x_1^*, x_2^*) . It is important to remember that this curve is not a trajectory. In fact the Poincaré intersection of any single trajectory is just a sequence of points as shown in Fig. 4.6. If a smooth curve is drawn on this kind of diagram, it represents the intersection points of an infinite family of trajectories, all of which are approaching (x_1^*, x_2^*) . Later we shall see cases in which such curves intersect. It is important to remember that this intersection does not violate the No-Intersection Theorem because the intersecting curves in this case are not themselves trajectories.

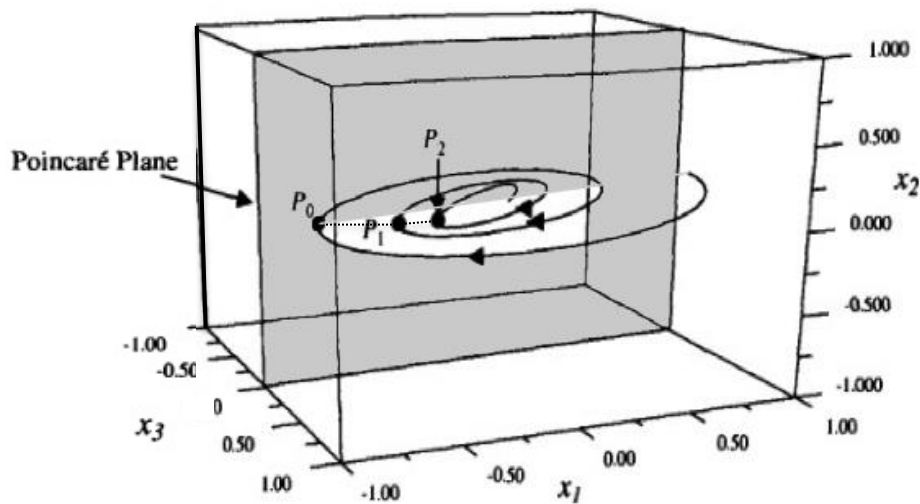
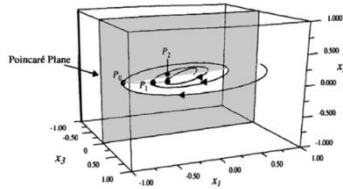


Fig. 4.6. The sequence of points P_0, P_1, P_2, \dots is the record of successive intersections of a single trajectory with the Poincaré plane (the plane with $x_3 = 0$) as the trajectory goes from $x_3 > 0$ to $x_3 < 0$.



MAPPA DI POINCARÉ' 2D PER LO STUDIO DELLA STABILITÀ' DEI CICLI LIMITE IN 3D

We now return to the general discussion of limit cycles. The stability of the limit cycle is determined by a generalization of the Poincaré multipliers introduced in the previous chapter. We assume that the uniqueness of the solutions to the equations used to describe the dynamical system entails the existence of a Poincaré map function (or in the present case, a pair of Poincaré map functions), which relate the coordinates of one point at which the trajectory crosses the Poincaré plane to the coordinates of the next (in time) crossing point. (Again we assume we have chosen a definite crossing sense; e.g., from top to bottom, or from left to right.) These functions take the form



$$\begin{aligned}x_1^{(n+1)} &= F_1(x_1^{(n)}, x_2^{(n)}) && \text{mappa di} \\x_2^{(n+1)} &= F_2(x_1^{(n)}, x_2^{(n)}) && \text{Poincaré 2 dim}\end{aligned} \quad (4.6-1)$$

where the parenthetical superscript indicates the crossing point number.

Here these Poincaré map functions have arisen from the consideration of a Poincaré section for trajectories arising from a set of differential equations. In Chapter 5, we shall consider such map functions as interesting models in their own right, independent of this particular heritage.

The fixed points of the Poincaré section are those points that satisfy

$$\begin{aligned}x_1^* &= F_1(x_1^*, x_2^*) \\x_2^* &= F_2(x_1^*, x_2^*)\end{aligned} \quad (4.6-2)$$

Each fixed point in the Poincaré section corresponds to a limit cycle in the full three-dimensional state space.

MAPPA DI POINCARÉ' 2D PER LO STUDIO DELLA STABILITÀ' DEI CICLI LIMITE IN 3D

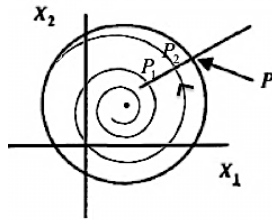
We can characterize the stability of these fixed points by finding the characteristic values of the associated Jacobian matrix of derivatives [sometimes called the Floquet matrix, after Gaston Floquet (1847–1920), a French mathematician who studied, among other things, the properties of differential equations with periodic terms]. This matrix is analogous to the Jacobian matrix used to determine the characteristic values of a fixed point in the full state space. The Jacobian matrix JM is given by

mappa 1 dim

$$d_2 = M d_1$$

$$d_{n+1} = M^n d_1$$

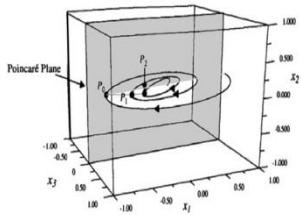
$$M = \left. \frac{dF}{dP} \right|_{P^*}$$



mappa 2 dim

$$JM = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{pmatrix} \longrightarrow \begin{matrix} M_1 \\ M_2 \end{matrix} \quad (4.6-3)$$

where the matrix is to be evaluated at the Poincaré map fixed point in question. The characteristic values of this matrix determine the stability of the limit cycle. A stable limit cycle attracts nearby trajectories, while an unstable limit cycle repels nearby trajectories. In principle, we can use the mathematical methods given in Chapter 3 to find these characteristic values. In practice, however, we most often cannot find these characteristic values explicitly, since, to do that, we would need to know the exact form of the Poincaré map function, and in most cases, we do not know that function. [In Chapter 5, we will examine some models that do give us the map function directly. However, for systems described by differential equations in state spaces of three (or more) dimensions, it is in general impossible to find the map functions.]



Stability of Limit Cycles

As we saw in two-dimensional systems, if the fixed point is to be stable and have trajectories in its neighborhood attracted to it, then the absolute value of each multiplier must be less than 1. [In state spaces with three or more dimensions, we can have $M < 0$; so the stability criterion is formulated using the absolute value of the multipliers.]

The types of limit cycles are

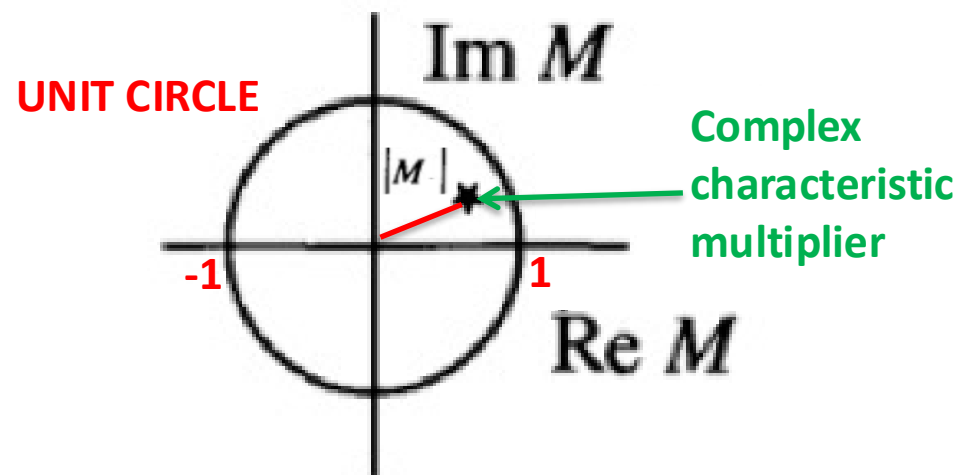
- I. **Stable limit cycle** (node for the Poincaré map)
- II. **Repelling limit cycle** (repellor for the Poincaré map)
- III. **Saddle cycle** (saddle point for the Poincaré map)

Table 4.2 lists the categories of characteristic multipliers, the associated Poincaré plane fixed points and the corresponding limit cycles for three-dimensional state spaces. (Compare this table to Table 3.4 for limit cycles in two-dimensional state spaces.)

Table 4.2
Characteristic Multipliers for Poincaré Sections
of Three-Dimensional State Spaces

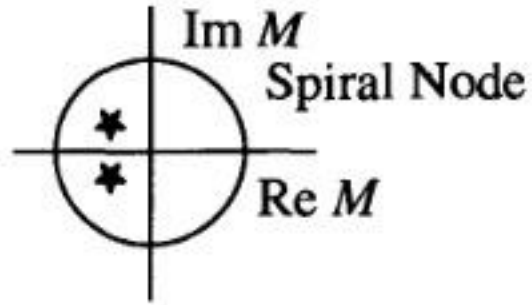
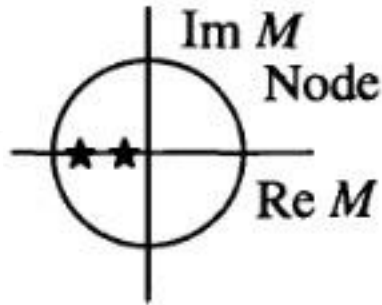
Type of Fixed Point	Characteristic Multiplier	Corresponding Cycle
Node	$ M_1 , M_2 < 1$	Limit Cycle
Repellor	$ M_1 , M_2 > 1$	Repelling Cycle
Saddle	$ M_1 < 1, M_2 > 1$	Saddle Cycle

Of course, the characteristic multipliers could also be complex numbers. Just as we saw for fixed points in a two-dimensional state space, the complex multipliers will form a complex-conjugate pair. In more graphic terms, the successive Poincaré intersection points associated with complex-valued multipliers rotate around the limit cycle intersection point as they approach or diverge from that point. Mathematically, the condition for stability is still the same: the absolute value of both multipliers must be less than 1 for a stable limit cycle. In terms of the corresponding Argand diagram (complex mathematical plane), both characteristic values must lie within a circle of unit radius (called the unit circle) for a stable limit cycle. See Fig. 4.7. As a control parameter is changed the values of the characteristic multipliers can change. If at least one of the characteristic multipliers crosses the unit circle, a bifurcation occurs. Some of these bifurcations will be discussed in the latter part of this chapter.



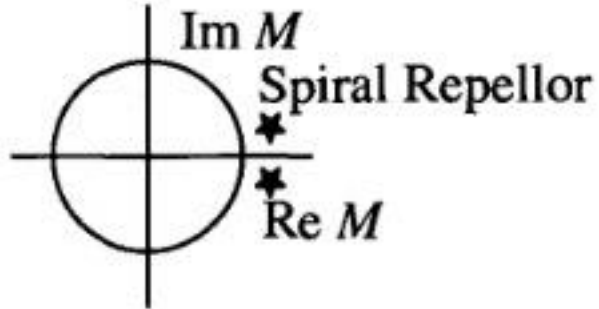
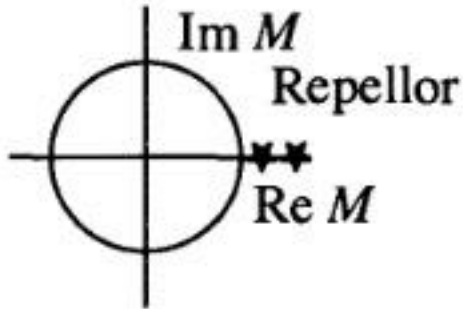
**Stable
Limit Cycles**

$$|M_1|, |M_2| < 1$$



**Repelling
Limit Cycles**

$$|M_1|, |M_2| > 1$$



**Saddle
Limit Cycles**

$$|M_1| < 1, |M_2| > 1$$

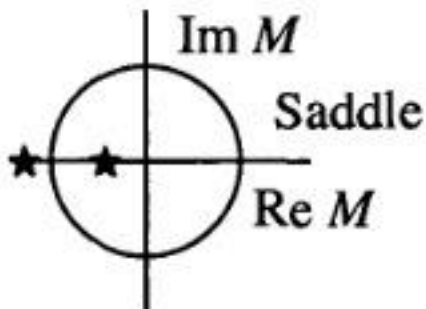
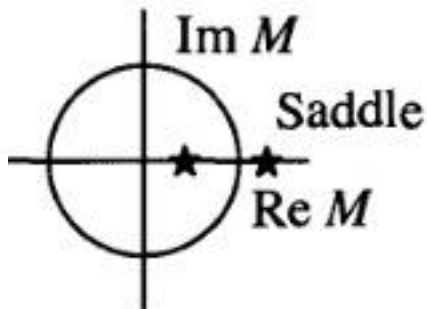


Fig. 4.7. Characteristic multipliers in the complex plane. If both multipliers lie within a circle of unit radius (the unit circle), then the corresponding limit cycle is stable. If one (or both) of the multipliers lies outside the unit circle, then the limit cycle is unstable.

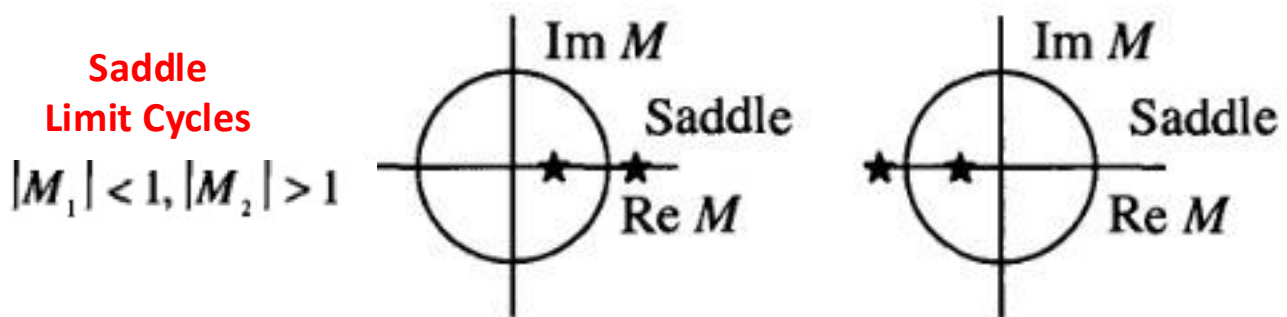
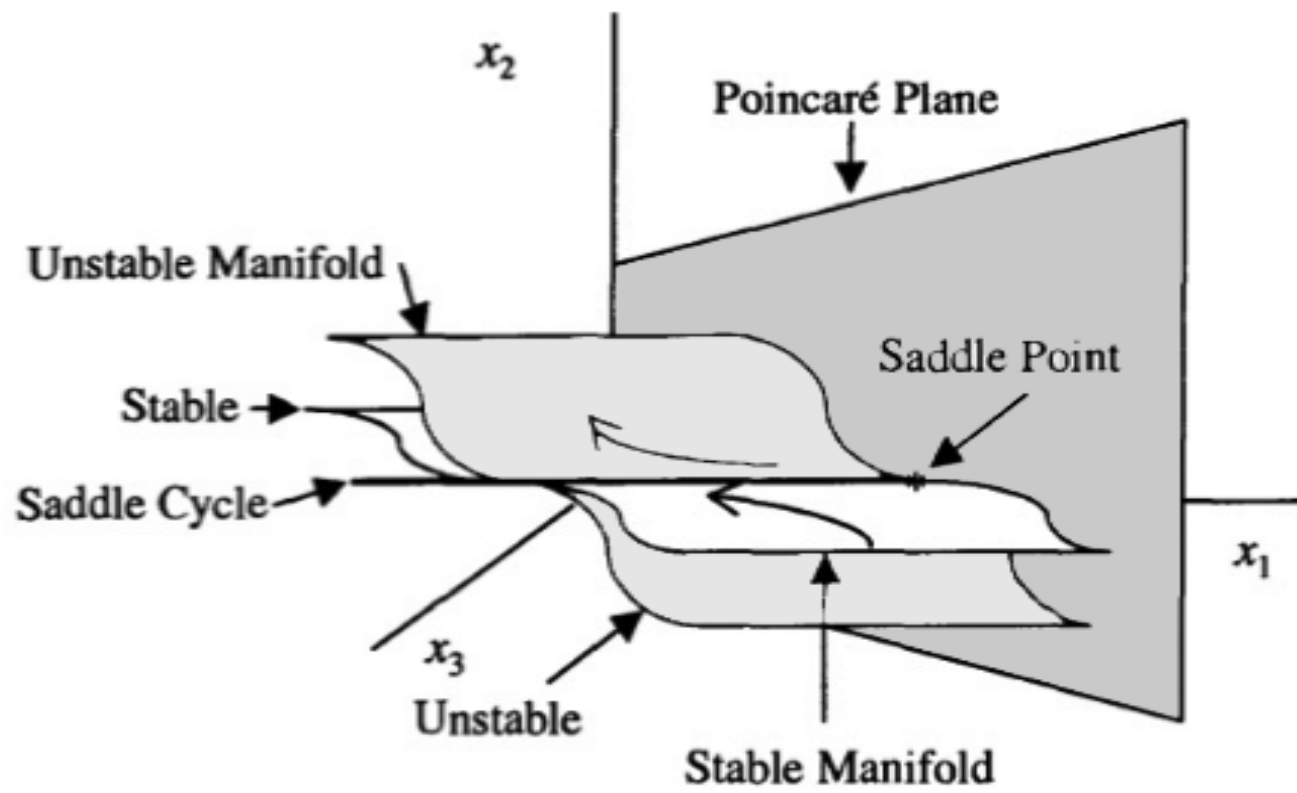


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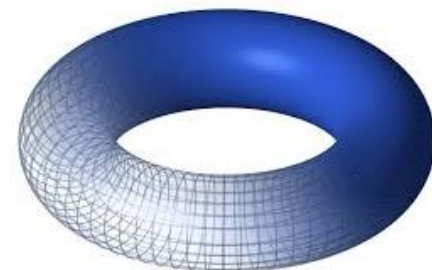
4.7 Quasi-Periodic Behavior (dim = 2)

For a three-dimensional state space, a new type of motion can occur, a type of motion not possible in one- or two-dimensional state spaces. This new type of motion is called quasi-periodic because it has two different frequencies associated with it; that is, it can be analyzed into two independent, periodic motions. For quasi-periodic motion, the trajectories are constrained to the surface of a torus in the three-dimensional state space. A mathematical description of this kind of motion is given by:

NOTA BENE: NON SONO EQUAZIONI DIFFERENZIALI!

Equazioni della traiettoria di un punto $(x_1(t), x_2(t)$ e $x_3(t))$ sull'attrattore quasiperiodico

$$\begin{cases} x_1 = (R + r \sin \omega_r t) \cos \omega_R t \\ x_2 = r \cos \omega_r t \\ x_3 = (R + r \sin \omega_r t) \sin \omega_R t \end{cases} \quad (4.7-1)$$



where the two angular frequencies are denoted by ω_R and ω_r . Geometrically, Eqs. (4.7-1) describe motion on the surface of a torus (with the center of the torus at the origin), whose large radius is R and whose cross-sectional radius is r . In general the torus (or doughnut-shape or the shape of the inner tube of a bicycle tire) will look something like Fig. 4.8. The frequency ω_R corresponds to the rate of rotation around the large circumference with a period $T_R = 2\pi/\omega_R$, while the frequency ω_r corresponds to the rate of rotation about the cross section with $T_r = 2\pi/\omega_r$. A general torus might have elliptical cross sections, but the ellipses can be made into circles by suitably rescaling the coordinate axes.

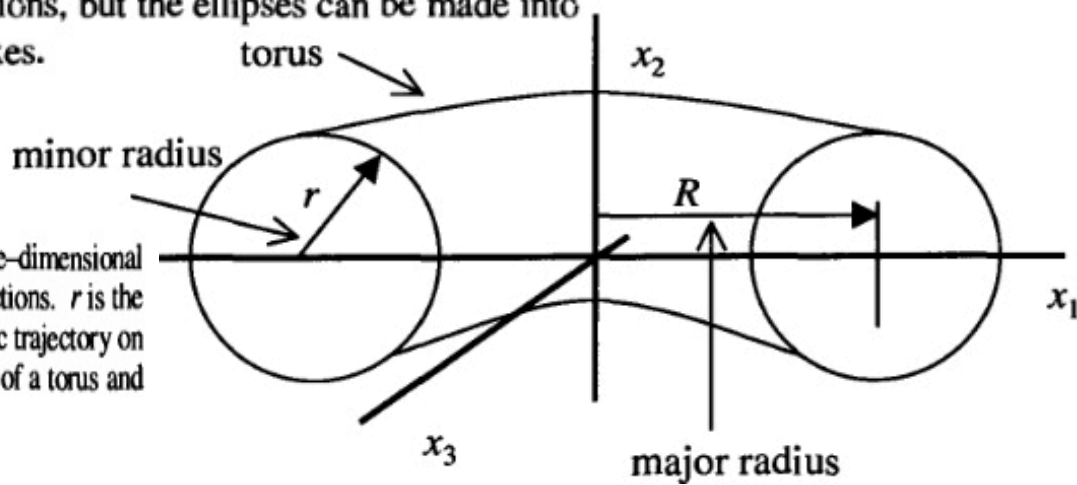


Fig. 4.8. Quasi-periodic trajectories roam over the surface of a torus in three-dimensional state space. Illustrated here is the special case of a torus with circular cross sections. r is the minor radius of the cross section. R is the major radius of the torus. A periodic trajectory on the surface of the torus would close on itself. On the right, a perspective view of a torus and a Poincaré plane.

The Poincaré section for this motion is generated by using a Poincaré plane that cuts through the torus. What the pattern of Poincaré map points looks like depends on the numerical relationship between the two frequencies as illustrated in Fig. 4.9. If the ratio of the two frequencies can be expressed as the ratio of two integers (that is, as a “rational fraction,” 14/17, for example), then the Poincaré section will consist of a finite number of points. This type of motion is often called frequency-locked motion because one of the frequencies is locked, often over a finite control parameter range, so that an integer multiple of one frequency is equal to another integer multiple of the other. (The terms *phase-locking* and *mode-locking* are also used to describe this behavior.)

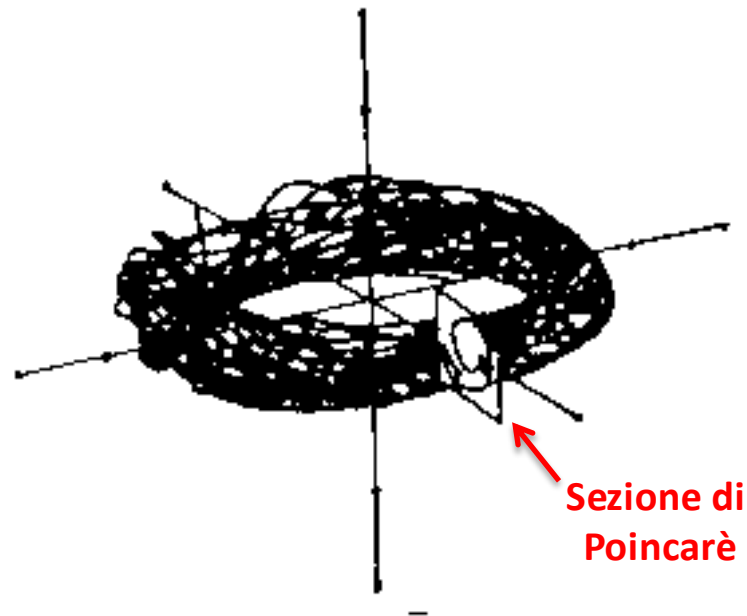
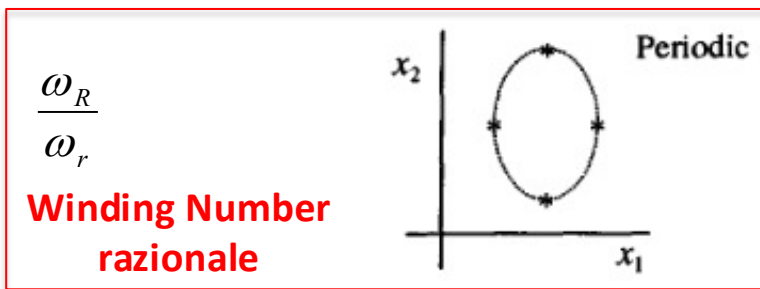
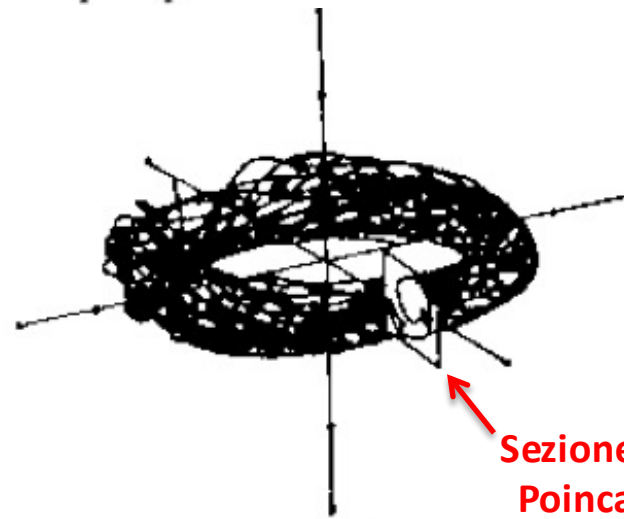
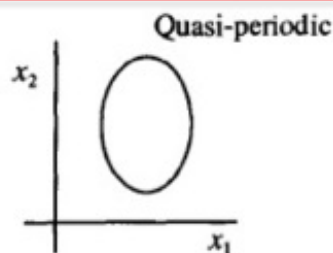
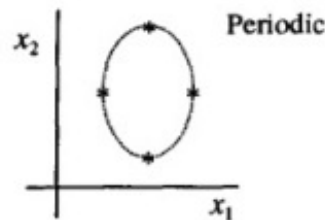


Fig. 4.9. A Poincaré section intersects a torus in three-dimensional state space. The diagram on the upper left shows the Poincaré map points for a two-frequency periodic system with a rational ratio of frequencies. The intersection points are indicated by asterisks. The diagram on the lower left is for quasi-periodic behavior. The ratio of frequencies is irrational, and eventually the intersection points fill in a curve (sometimes called a “drift ring”) in the Poincaré plane.

If the ratio of frequencies cannot be expressed as a ratio of integers, then the ratio is called "irrational" (in the mathematical, not the psychological sense). For the irrational case, the Poincaré map points will eventually fill in a continuous curve in the Poincaré plane, and the motion is said to be quasi-periodic because the motion never exactly repeats itself. (Russian mathematicians call this *conditionally periodic*. See, for example, [Arnold, 1983]. The term *almost periodic* is also used in the mathematical literature.)

In the quasi-periodic case the motion, strictly speaking, never exactly repeats itself (hence, the modifier *quasi*), but the motion is not chaotic; it is composed of two (or more) periodic components, whose presence could be made known by measuring the frequency spectrum (Fourier power spectrum) of the motion. We should point out that detecting the difference between quasi-periodic motion and motion with a rational ratio of frequencies, when the integers are large, is a delicate question. Whether a given experiment can distinguish the two cases depends on the resolution of the experimental equipment. As we shall see later, the behavior of the system can switch abruptly back and forth between the two cases as a parameter of the system is varied. The important point is that the attractor for the system is a two-dimensional surface of the torus for quasi-periodic behavior.

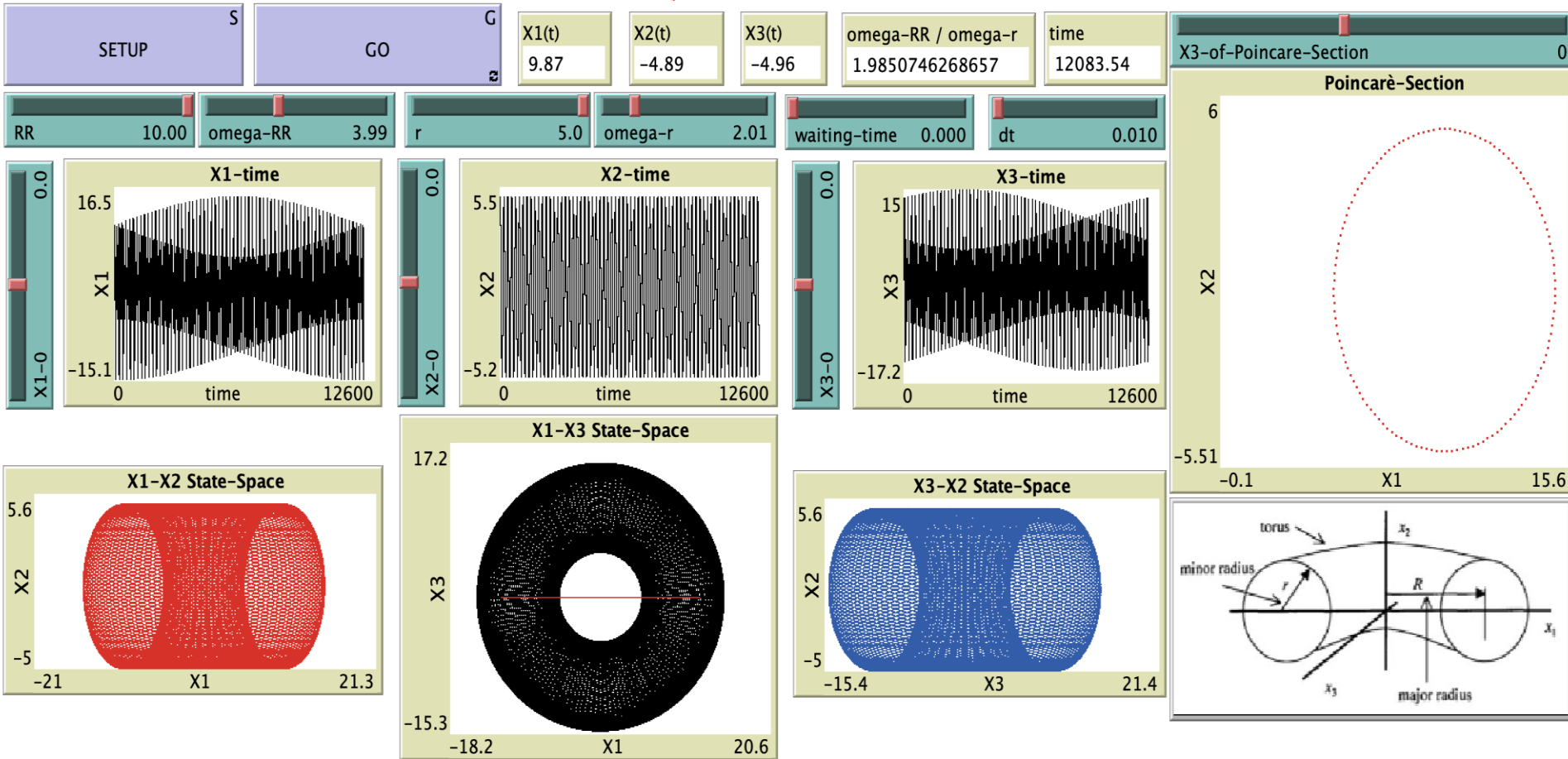


**Winding Number
irrazionale**

$$\frac{\omega_R}{\omega_r}$$

quasi-periodicity.nlogo

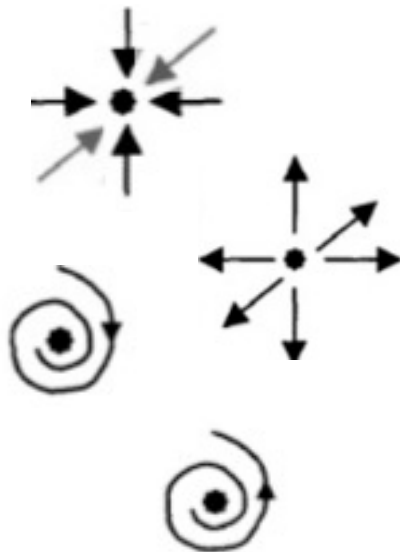
QUASI-PERIODICITY



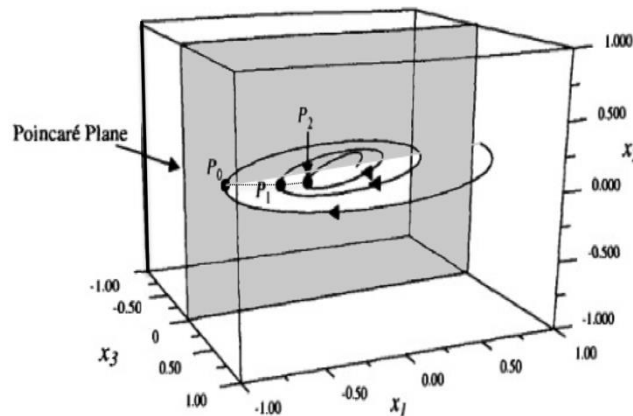
We have now seen the full panoply of regular (nonchaotic) attractors: fixed points (dimension 0), limit cycles (dimension 1), and quasi-periodic attractors (dimension 2 or more). We are ready to begin the discussion of how these attractors can change into chaotic attractors.

We will give only a brief discussion of the period-doubling, quasi-periodic, and intermittency routes. These will be discussed in detail in Chapters 5, 6, and 7, respectively. A discussion of crises will be found in Chapter 7. As we shall see, the chaotic transient route is more complicated to describe because it requires a knowledge of what trajectories are doing over a range of state space. We can no longer focus our attention locally on just a single fixed point or limit cycle.

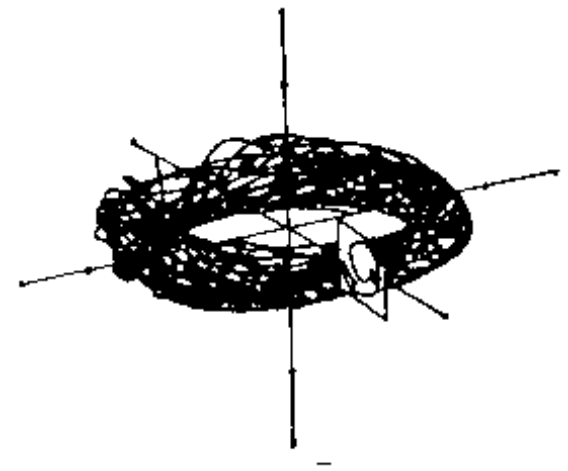
fixed points (dim.0)



limit cycles (dim.1)

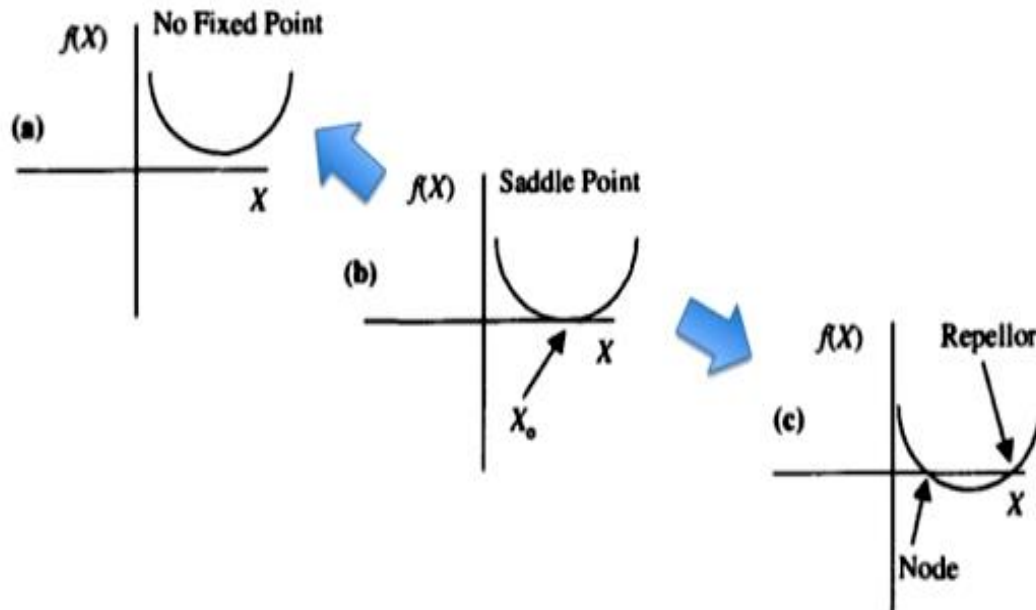


quasiperiodic attractors (dim.2)

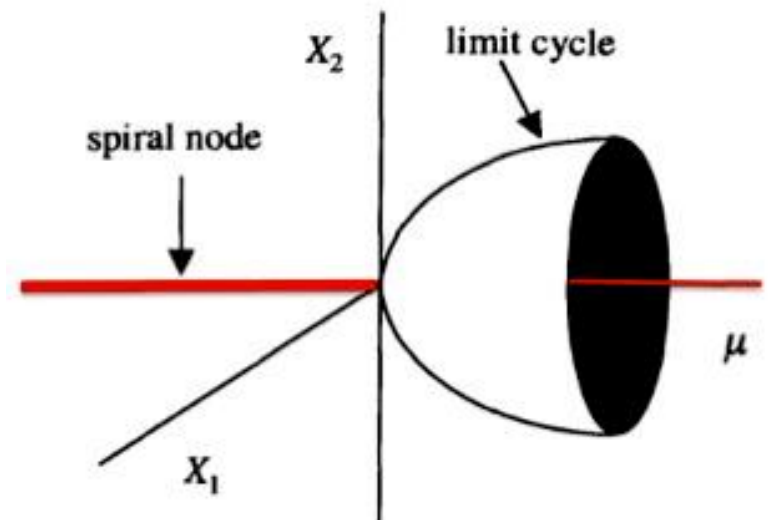


Biforcazioni

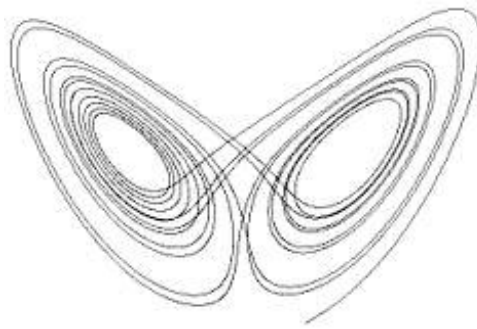
1D-2D: Repellor-Node (or Saddle-Node or Tangent) Bifurcation



2D: Hopf Bifurcation

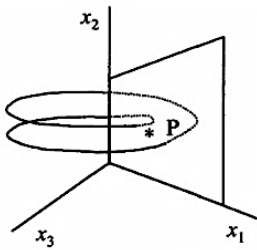
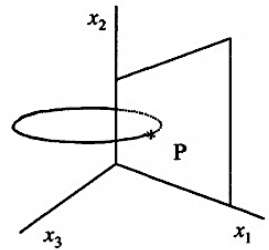


3D: . . . (Rotte verso il Caos...)

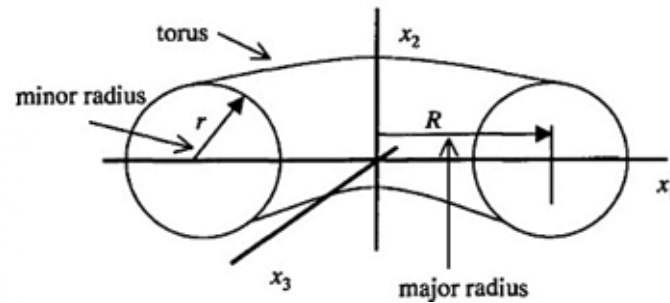


Rotte verso il CAOS...

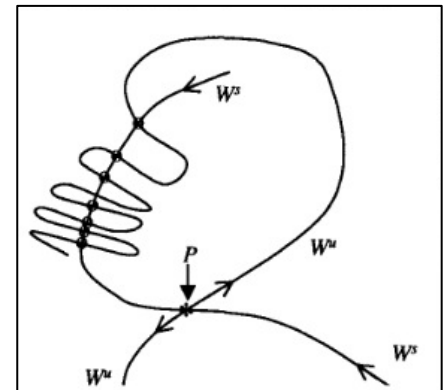
I : Period-Doubling



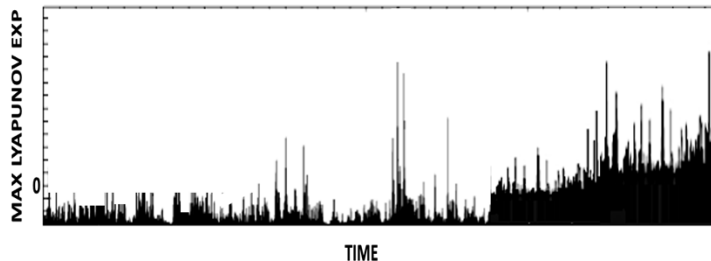
II : Quasi-Periodicity

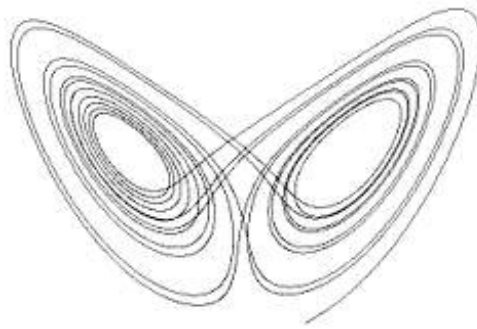


IV : Chaotic Transient and Homoclinic Orbits



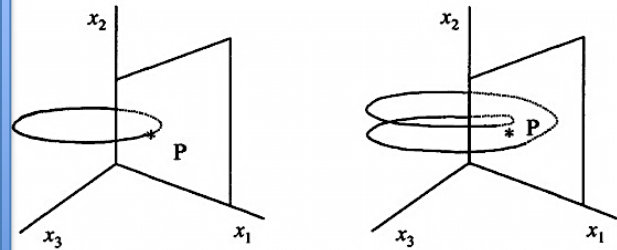
III : Intermittency and Crises





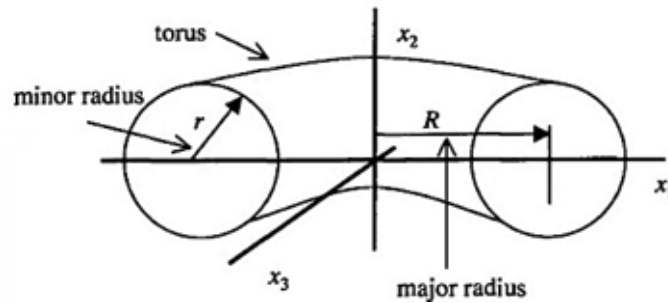
Rotte verso il CAOS...

I : Period-Doubling

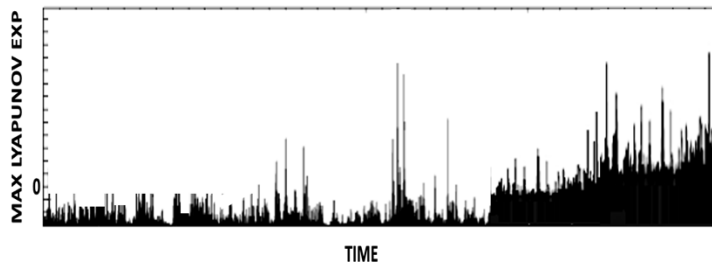


Local Bifurcations

II : Quasi-Periodicity

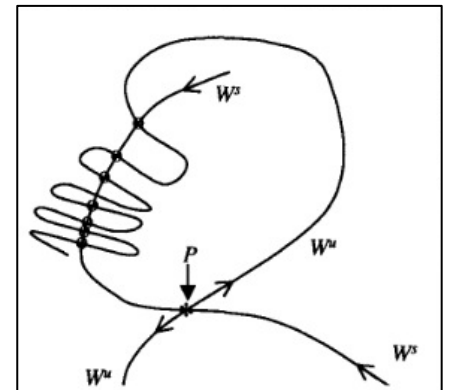


III : Intermittency and Crises



Global Bifurcations

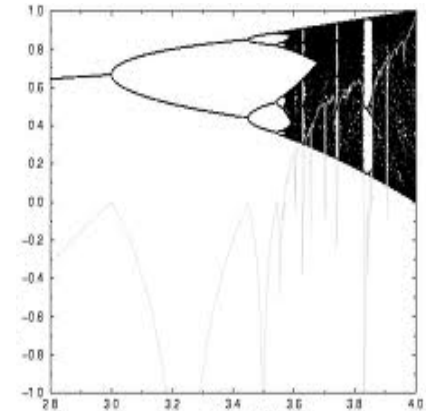
IV : Chaotic Transient and Homoclinic Orbits



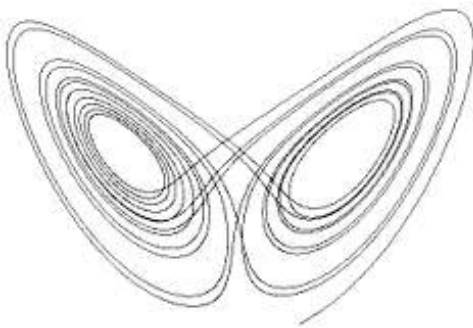
Rotte verso il CAOS

When the parameters of a system are changed, chaotic behavior may appear and disappear in several different ways, even for the same dynamical system: We may have several routes to chaos. These routes can be put into two broad categories with several subdivisions within each category. One category includes sequences of bifurcations involving limit cycles (or equivalently, fixed points of the associated Poincaré map). (The period-doubling sequence in Chapter 1 belongs to this category.) The other category involves changes in trajectories associated with several fixed points or limit cycles. Since these changes involve the properties of trajectories ranging over a significant volume of state space, these changes are called “global” bifurcations (in contrast to “local” bifurcations associated with changes in individual fixed points).

Local Bifurcations



Global Bifurcations



Rather sudden changes from regular to chaotic behavior, such as we have seen with the Lorenz model in Chapter 1, are characteristic of these global bifurcations. Although the nature of the long-time attractor changes suddenly as a parameter is varied, these sudden changes are often heralded by chaotic transients. In a chaotic transient, the system's trajectory wanders through state space, in an apparently chaotic fashion. Eventually, the trajectory approaches a regular, periodic attractor. As the control parameter is changed, the chaotic transient lasts longer and longer until finally the asymptotic behavior is itself chaotic.

The questions we want to address are the following: How does this complicated chaotic behavior develop? How does the system evolve from regular, periodic behavior to chaotic behavior? What changes in the fixed points and in trajectories in state space give rise to these changes in behavior?

I

4.8 The Routes to Chaos I: Period-Doubling

As we discussed earlier, the period-doubling route begins with limit cycle behavior of the system. This limit cycle, of course, may have been “born” from a bifurcation involving a node or other fixed point, but we need not worry about that now. As some control parameter changes, this limit cycle becomes unstable. Again this event is best viewed in the corresponding Poincaré section. Let us assume that the periodic limit cycle generates a single point in the Poincaré section. If the limit cycle becomes unstable by having one of its characteristic multipliers become more negative than -1 (which, of course, means $|M| > 1$), then, in many situations, the new motion remains periodic but has a period twice as long as the period of the original motion. In the Poincaré section, this new limit cycle exhibits two points, one on each side of the original Poincaré section point (see Fig. 4.10).

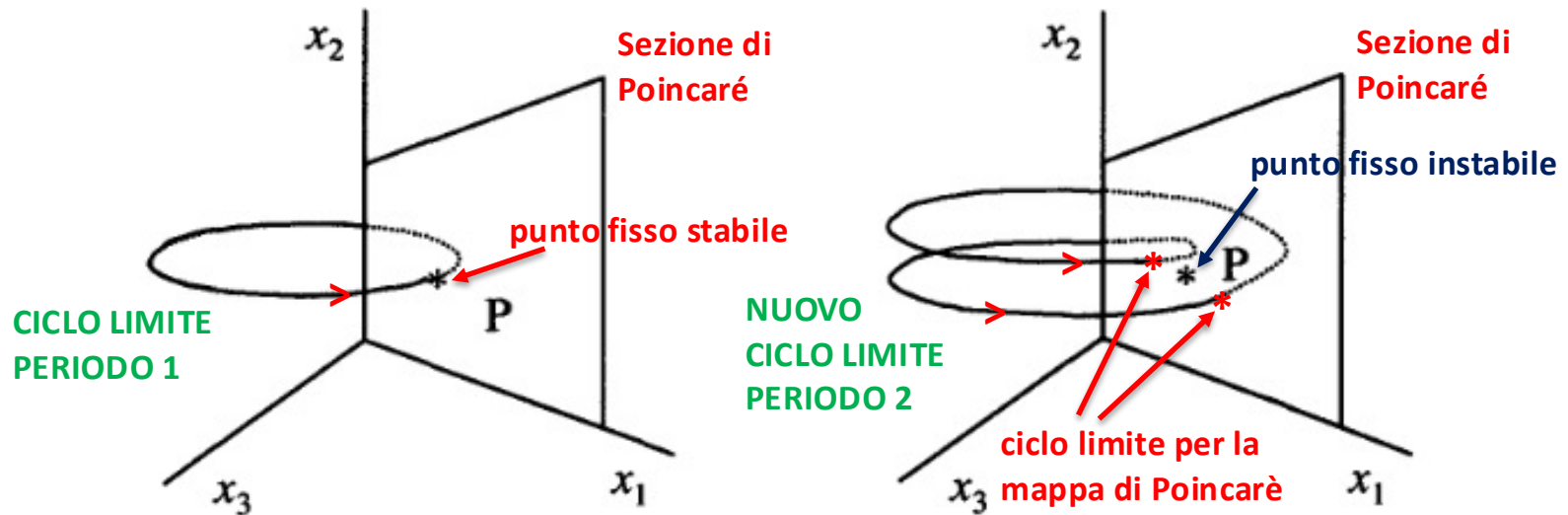
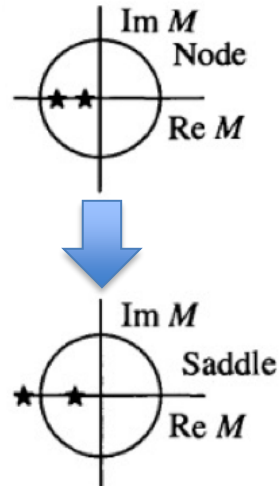
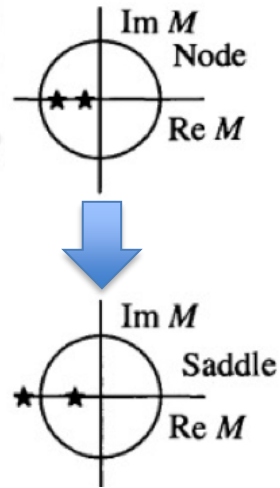


Fig. 4.10. The Poincaré section of a trajectory that has undergone a period-doubling bifurcation. On the left is the original periodic trajectory, which intersects the Poincaré plane in one point. On the right is the period-doubled trajectory, which intersects the Poincaré plane in two points, one on each side of the original intersection point.

This alternation of intersection points is related to the characteristic multiplier associated with the original limit cycle, which has gotten more negative than -1 . Since $|M| > 1$, the trajectory's map points are now being "repelled" by the original map point. The minus sign tells us that they alternate from one side to the other, as we can see formally from Eq. (3.16-6). This type of bifurcation is also called a flip bifurcation because the newly born trajectory flips back and forth from one side of the original trajectory to the other.

As the control parameter is changed further, this period-two limit cycle may become unstable and give birth to a period-four cycle with four Poincaré intersection points. Chapter 5 will examine in detail how, when, and where this sequence occurs. The period-doubling process may continue until the period becomes infinite; that is, the trajectory never repeats itself. The trajectory is then chaotic.



mappa 1 dim

$$d_{n+1} = M^n d_1$$

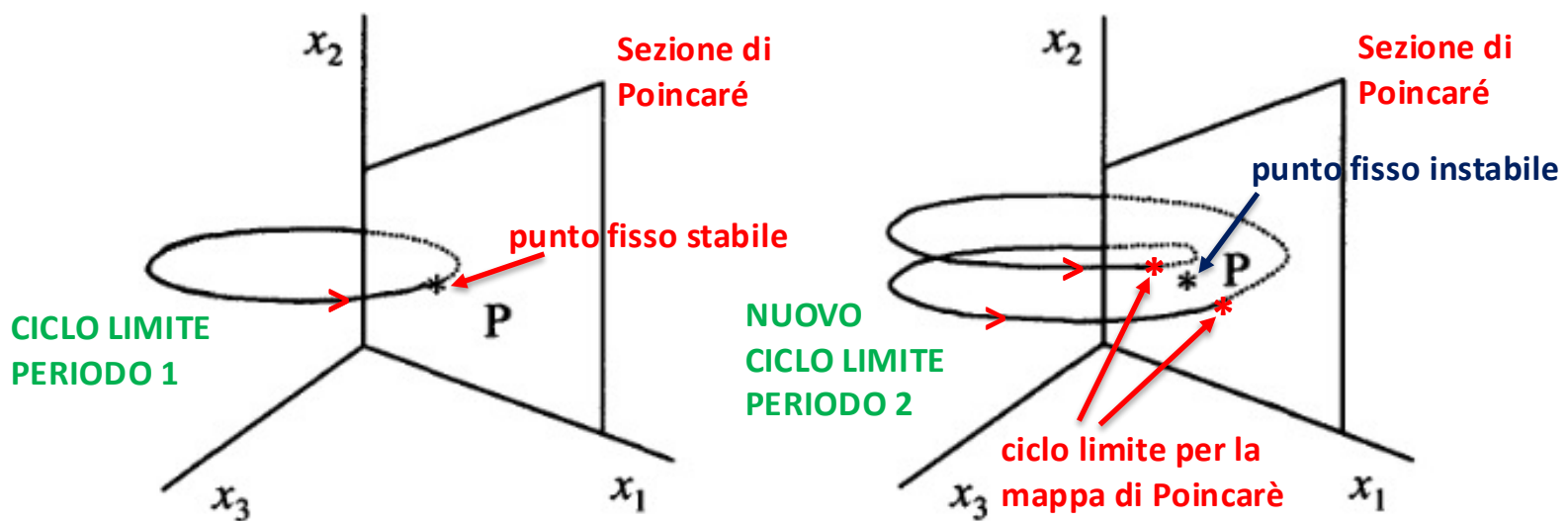


Fig. 4.10. The Poincaré section of a trajectory that has undergone a period-doubling bifurcation. On the left is the original periodic trajectory, which intersects the Poincaré plane in one point. On the right is the period-doubled trajectory, which intersects the Poincaré plane in two points, one on each side of the original intersection point.

II

4.9 The Routes to Chaos II: Quasi-Periodicity

In the quasi-periodic scenario, the system begins again with a limit cycle trajectory. As a control parameter is changed, a second periodicity appears in the behavior of the system. This bifurcation event is a generalization of the Hopf bifurcation discussed in Chapter 3; so, it is also called a Hopf bifurcation. In terms of the characteristic multipliers, the Hopf bifurcation is marked by having the two complex-conjugate multipliers cross the unit circle simultaneously.

If the ratio of the period of the second type of motion to the period of the first is not a rational ratio, then we say, as described previously, that the motion is quasi-periodic. Under some circumstances, if the control parameter is changed further, the motion becomes chaotic. This route is sometimes called the Ruelle–Takens scenario after D. Ruelle and F. Takens, who in 1971 first suggested the theoretical possibility of this route. The main point here is that you might expect, at first thought, to see a long sequence of different frequencies come in as the control parameter is changed, much like the infinite sequence of period-doublings described in the previous section. (In 1944 the Russian physicist L. Landau had proposed such an infinite sequence of frequencies as a mechanism for producing fluid turbulence. [Landau and Lifshitz, 1959].) However, at least in some cases, the system becomes chaotic instead of introducing a third distinct frequency for its motion. This scenario will be discussed in Chapter 6.

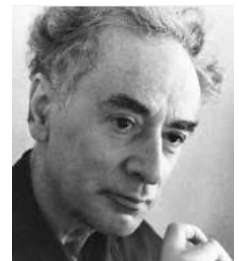
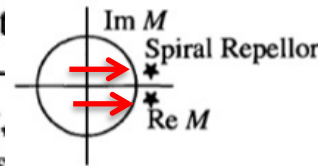
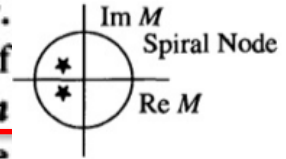
Historically, the experimental evidence for the quasi-periodic route to chaos (GOS75) played an important role in alerting the community of scientists to the utility of many of the newly emerging ideas in nonlinear dynamics. During the late 1970s and early 1980s there were many theoretical conjectures about the necessity of the transition from two-frequency quasi-periodic behavior to chaos. More recent work (see for example, BAT88) has shown that systems with significant spatial extent and with more degrees of freedom can have quasi-periodic behavior with three or more frequencies before becoming chaotic.



David Ruelle
(1935)



Floris Takens
(1940-2010)



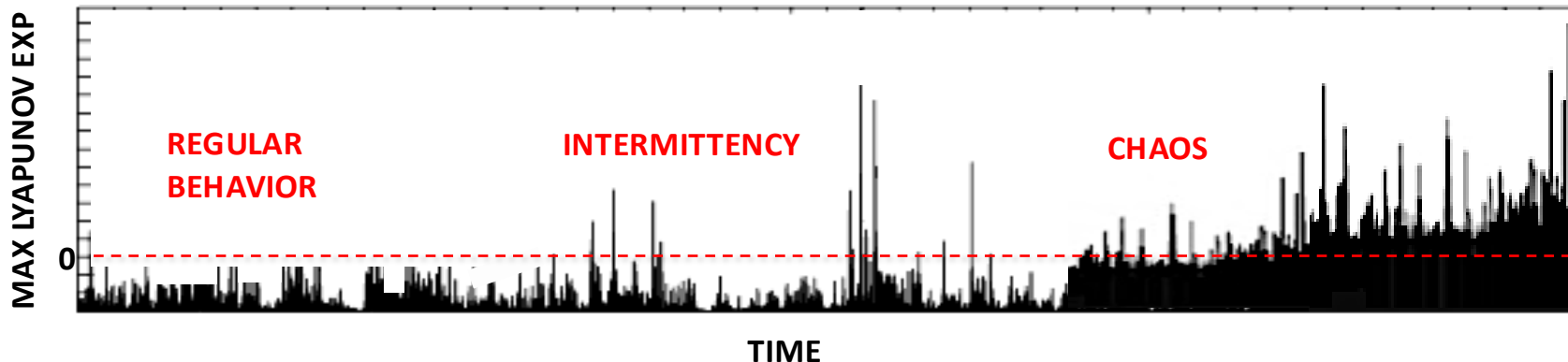
Lev Landau
(1908-1968)

III

4.10 The Routes to Chaos III: Intermittency and Crises

Chapter 7 contains a detailed discussion of intermittency and crises; so, we will give only the briefest description here. The **intermittency** route to chaos is characterized by dynamics with irregularly occurring bursts of chaotic behavior interspersed with intervals of apparently periodic behavior. As some control parameter of the system is changed, the chaotic bursts become longer and occur more frequently until, eventually, the entire time record is chaotic.

A **crisis** is a bifurcation event in which a chaotic attractor and its basin of attraction suddenly disappear or suddenly change in size as some control parameter is adjusted. Alternatively, if the parameter is changed in the opposite direction, the chaotic attractor can suddenly appear “out of the blue” or the size of the attractor can suddenly be reduced. As we shall see in Chapter 7, a crisis event involves the interaction between a chaotic attractor and an unstable fixed point or an unstable limit cycle.



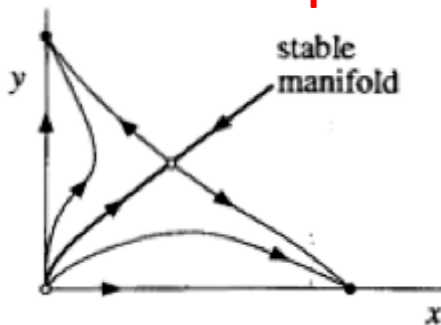
IV

4.11 The Routes to Chaos IV: Chaotic Transients and Homoclinic Orbits

In our second broad category of routes to chaos, the global bifurcation category, the chaotic transient route is the most important for systems modeled by sets of ordinary differential equations. Although the chaotic transient route to chaos was one of the first to be recognized in a model of a physical system (the Lorenz model), the theory of this scenario is, in some ways, less well developed than the theory for period-doubling, quasi-periodicity, and intermittency. This lack of development is due to the fact that this transition to chaos is not (usually) marked by any change in the fixed points of the system or the fixed points of a Poincaré section. The transition is due to the interaction of trajectories with various unstable fixed points and cycles in the state space. The common features are the so-called homoclinic orbits and their cousins, heteroclinic orbits. These special orbits may suddenly appear as a control parameter is changed. More importantly, these orbits strongly influence the nature of other trajectories passing near them.

What is a homoclinic orbit? To answer this question, we need to consider saddle cycles in a three-dimensional state space. (These ideas carry over in a straightforward fashion to higher-dimensional state spaces.) You should recall that saddle points and saddle cycles and, in particular, their in-sets and out-sets serve to organize the state space. That is, the in-sets and out-sets serve as “boundaries” between different parts of the state space and all trajectories must respect those boundaries. We will focus our attention on a saddle point in the Poincaré section of the state space. This saddle point corresponds to a saddle cycle in the original three-dimensional state space (see Fig. 4.11). We can consider the saddle cycle to be the intersection between two surfaces: One surface is the in-set (that is, the stable manifold) associated with the cycle. The other surface is the out-set (unstable manifold) associated with the cycle. Trajectories on the in-set approach the saddle cycle as time goes on. Trajectories on the out-set diverge from the cycle as time goes on. Trajectories that are near to, but not on, the in-set will first approach the cycle and then be repelled roughly along the out-set (unstable manifold).

2 dim: saddle point



3 dim: saddle cycle

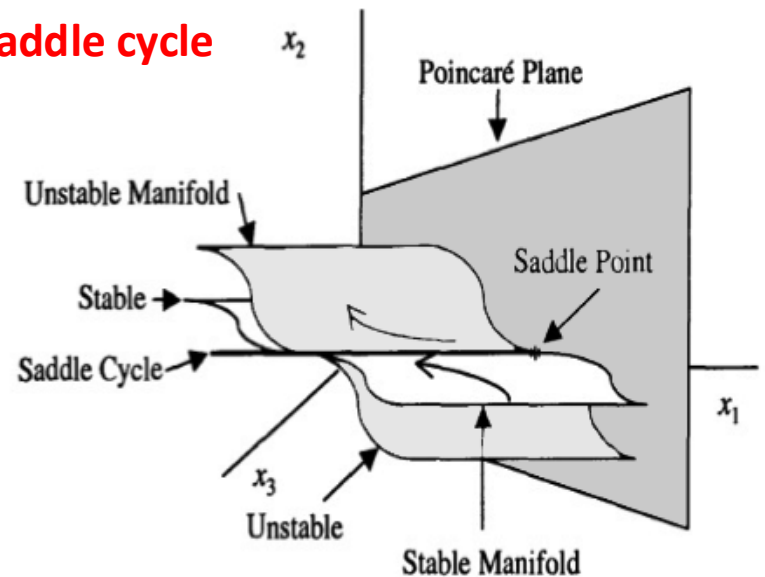


Fig. 4.11. A saddle cycle in a three-dimensional state space. The stable and unstable manifolds are surfaces that intersect at the saddle cycle. Where the saddle cycle intersects a Poincaré plane we have a saddle point for the Poincaré map function. A portion of one trajectory, repelled by the saddle cycle, is shown on the unstable manifold, and a portion of another trajectory, approaching the saddle cycle, is shown on the stable manifold.

Fig. 4.12 shows the equivalent Poincaré section with a saddle point P where the saddle cycle intersects the plane. We have sketched in some curves to indicate schematically where the in-set and out-set surfaces cut the Poincaré plane. These curves, labeled $W^s(P)$ and $W^u(P)$, are called the stable and unstable manifolds of the saddle point P . Since the in-set and out-set of a saddle cycle are generally two-dimensional surfaces (in the original three-dimensional state space), the intersection of one of these surfaces with the Poincaré plane forms a curve. It is crucial to realize that these curves are not trajectories. For example, if we pick a point s_0 on $W^s(P)$, a point at which some trajectory intersects the Poincaré plane, then the Poincaré map function F gives us s_1 , the coordinates of the point at which the trajectory next intersects the plane. From s_1 , we can find s_2 , and so on. The sequence of points lies along the curve labeled $W^s(P)$ and approaches P as $n \rightarrow \infty$. Similarly, if u_0 is a point along $W^u(P)$, then $F(u_0) = u_1$, $F(u_1) = u_2$, and so on, generates a series of points that diverges from P along $W^u(P)$. If we apply the inverse of the Poincaré map function $F^{(-1)}$ to u_0 , we generate a sequence of points u_{-1} , u_{-2} , and so on, that approaches P as $n \rightarrow -\infty$. The curves drawn in Fig. 4.12 represent the totality of such sequences of points taken with infinitely many starting points. For any one starting point, however, the sequence jumps along W^s or W^u , it does not move smoothly like a point on a trajectory in the original state space.

Sezione di Poincarè

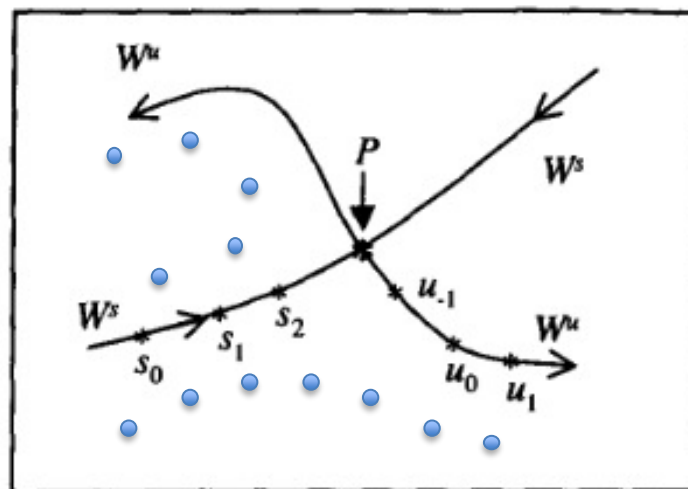


Fig. 4.12. Point P is a saddle point in a Poincaré section. It corresponds to a saddle cycle in the full three-dimensional state space. The intersection of the in-set of the saddle cycle with the Poincaré plane generates the curve labeled W^s . The intersection of the out-set of the saddle cycle with the Poincaré plane generates curve W^u .

As a control parameter is changed, it is possible for $W^s(P)$ and $W^u(P)$ to approach each other and in fact to intersect, say, at some point q . If this intersection occurs, we say that we have a homoclinic intersection at q , and the point q is called a homoclinic (intersection) point. It is also possible for the unstable manifold of one saddle point to intersect the stable manifold from some other saddle point. In that case we say we have a heteroclinic intersection. Other heteroclinic combinations are possible. For example, we could have the intersection of the unstable manifold surface of an index-2 saddle point and the stable manifold of a saddle cycle. (For a nice visual catalog of the possible kinds of intersections, see [Abraham and Shaw, 1992][Abraham, Abraham, and Shaw, 1996].) For now we will concentrate on homoclinic intersections.

We now come to an important and crucial theorem:

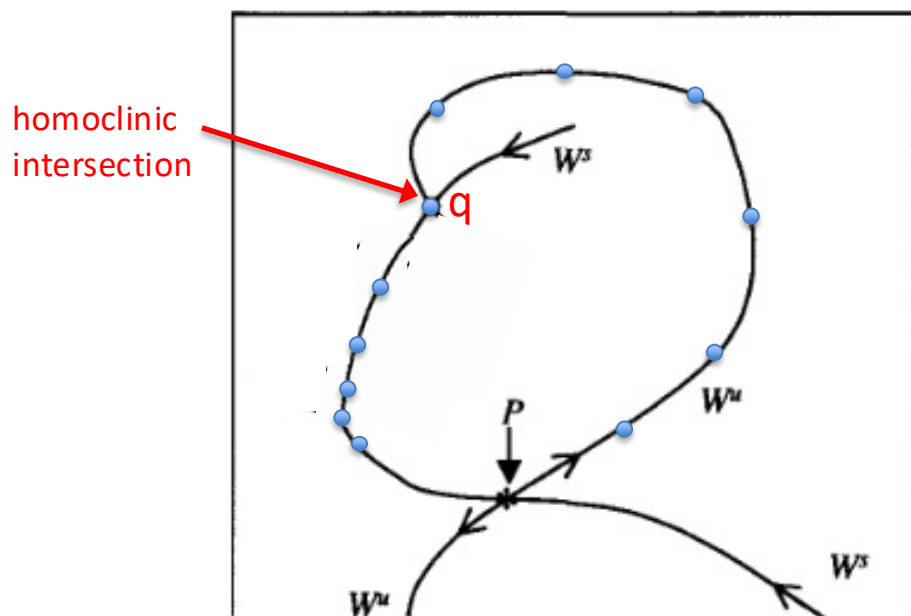


Fig. 4.13. A homoclinic tangle results from the homoclinic intersection of the unstable manifold $W^u(P)$ with the stable manifold $W^s(P)$ of the saddle point P . Each of the circled points is a homoclinic (intersection) point. For clarity's sake, only a portion of the tangle is shown.

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If the in-sets and out-sets of a saddle point in the Poincaré section of a dynamical system intersect at one homoclinic intersection point q_0 , then there must be an infinite number of homoclinic intersections.

To prove this statement, we consider the result of applying the mapping function F to q_0 . We get another point q_1 . Since q_0 belongs to both W^s and W^u , so must q_1 , since we have argued in the previous paragraphs that applying F to a point on W^s or W^u generates another point on W^s or W^u . By continuing to apply F to this sequence of points, we generate an infinite number of homoclinic points. Fig. 4.13 shows part of the resulting homoclinic tangle, which must result from the homoclinic intersections.

Please note that the smooth curves drawn in Fig. 4.13 are not individual trajectories. (We cannot violate the No-Intersection Theorem!) The smooth curves are generated by taking infinitely many starting points on W^s and W^u . Only those trajectories that hit one of the homoclinic points will hit (some of) the other homoclinic points.

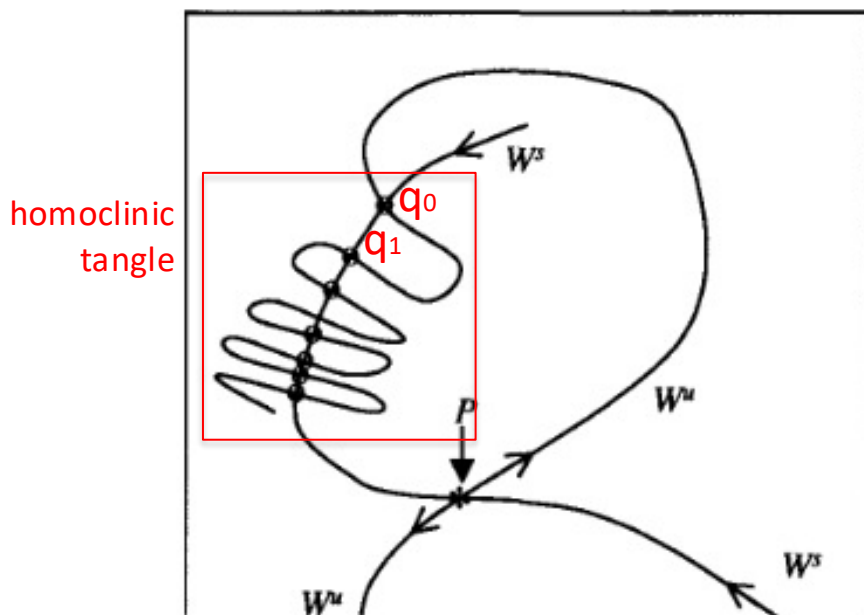
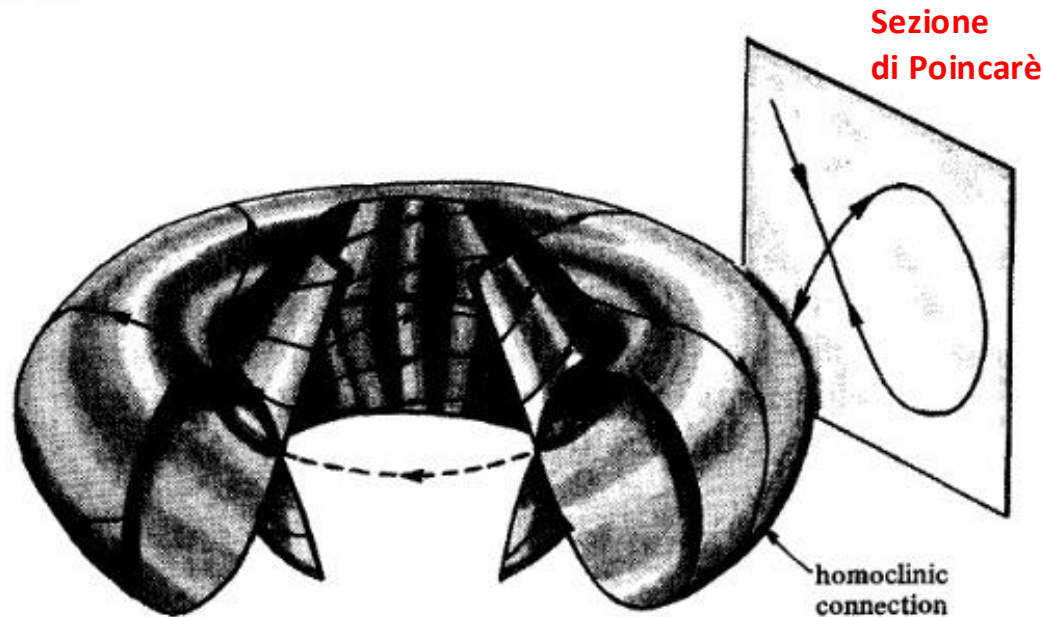
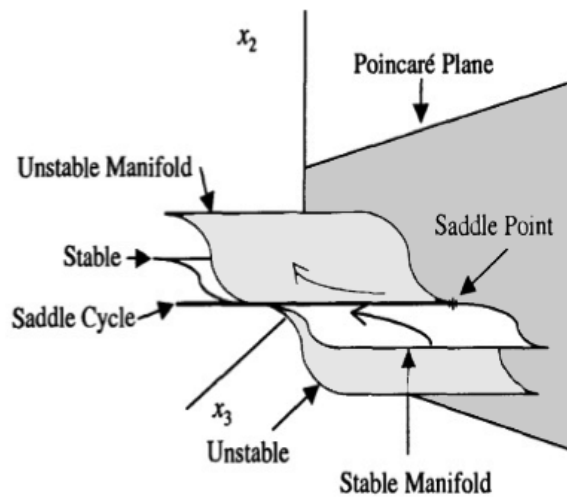


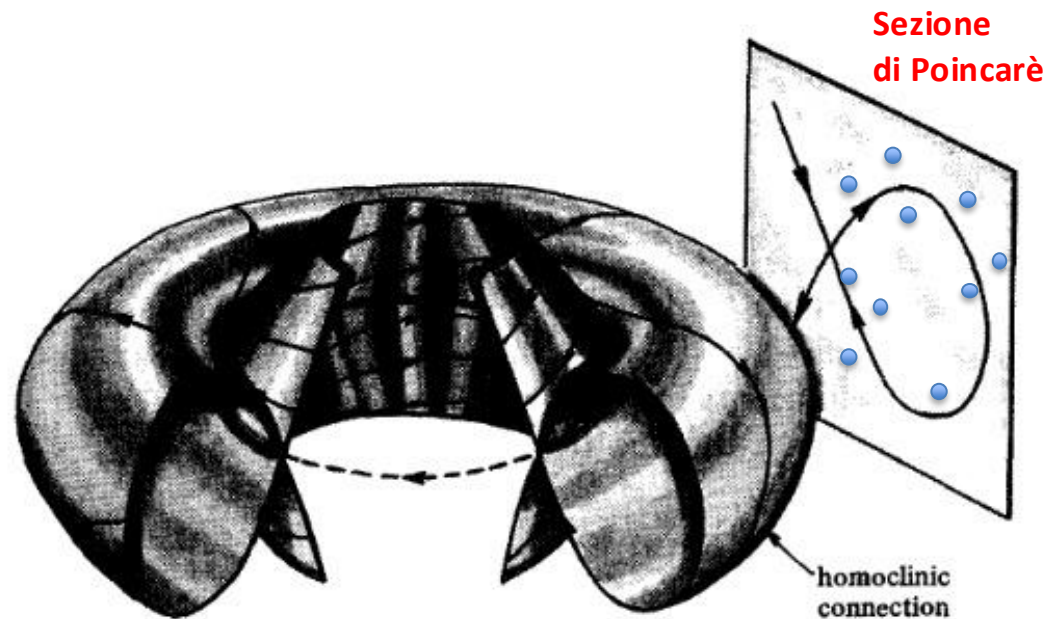
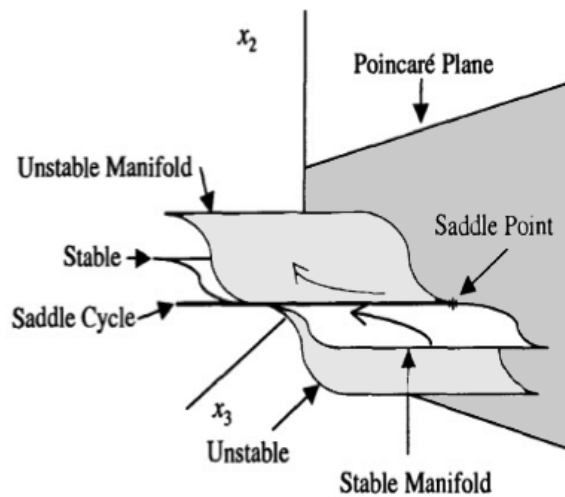
Fig. 4.13. A homoclinic tangle results from the homoclinic intersection of the unstable manifold $W^u(P)$ with the stable manifold $W^s(P)$ of the saddle point P . Each of the circled points is a homoclinic (intersection) point. For clarity's sake, only a portion of the tangle is shown.

What is the dynamical significance of a homoclinic point and the related homoclinic tangle? If we now shift our attention back to the original three-dimensional state space, we see that a homoclinic point in the Poincaré section corresponds to a continuous trajectory in the original state space. When a homoclinic intersection occurs, one trajectory on the unstable manifold joins another trajectory on the stable manifold to form a single new trajectory whose Poincaré intersection points are the homoclinic points described earlier. (To help visualize this process, recall that the in-set and out-set of a saddle cycle in the three-dimensional state space are, in general, two-dimensional surfaces.) This new trajectory connects the saddle point to itself and hence is called a homoclinic trajectory or homoclinic orbit and is said to form a homoclinic connection. As our previous theorem states, this homoclinic trajectory must intersect the Poincaré plane an infinite number of times.



(a)

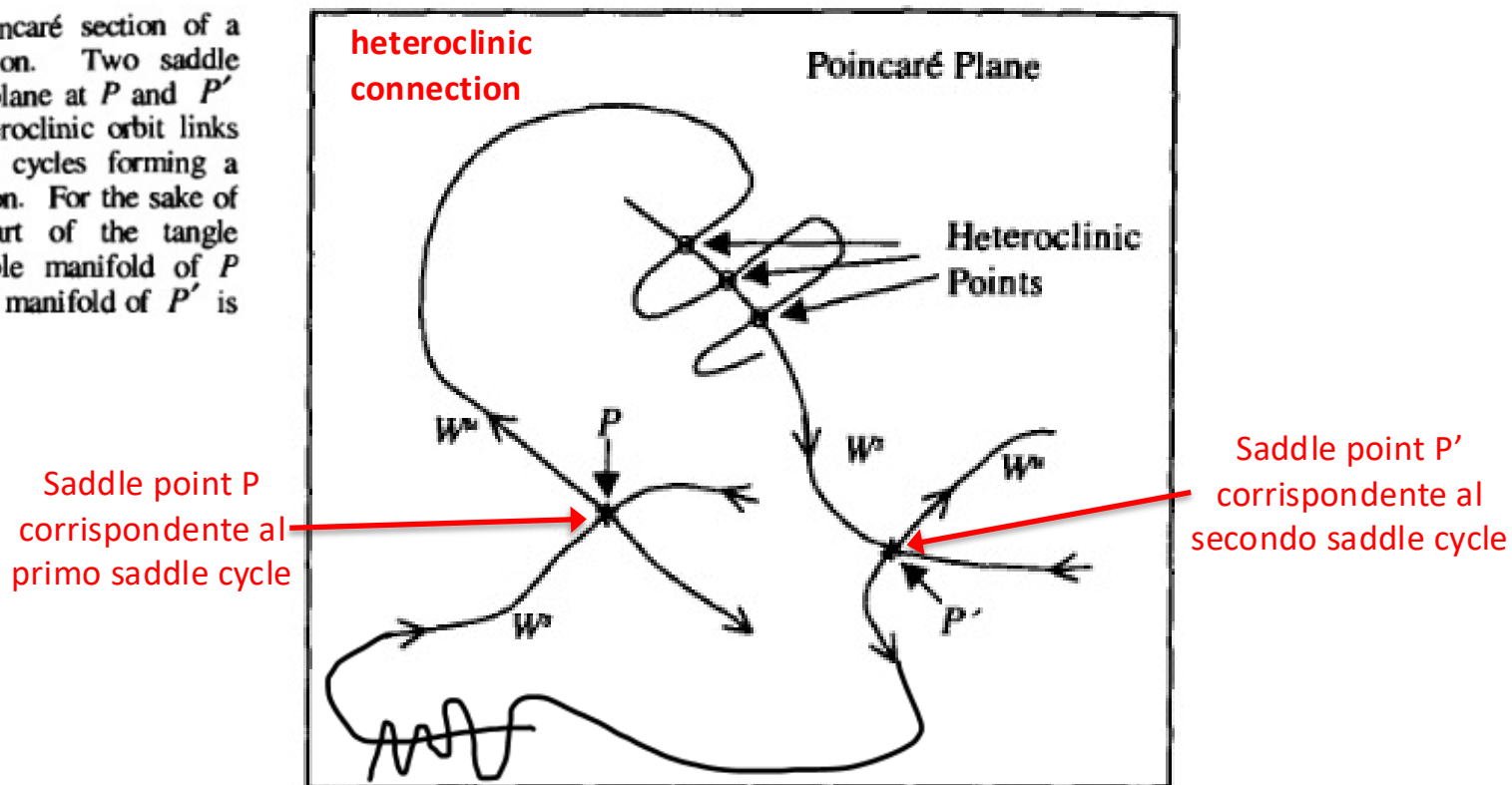
How does a homoclinic orbit lead to chaotic behavior? To understand this, we need to consider other trajectories that come near the saddle point of the Poincaré section. Generally speaking, the trajectories approach the saddle point close to (but not on) the in-set (stable manifold), but they are then forced away from the saddle point near the out-set (unstable manifold). After a homoclinic tangle has developed, a trajectory will be pushed away from the saddle point by the out-set part of the tangle, but it will be pulled back by the in-set part. It is easy to see that the homoclinic tangle can lead to trajectories that seem to wander randomly around the state space region near the saddle point.



(a)

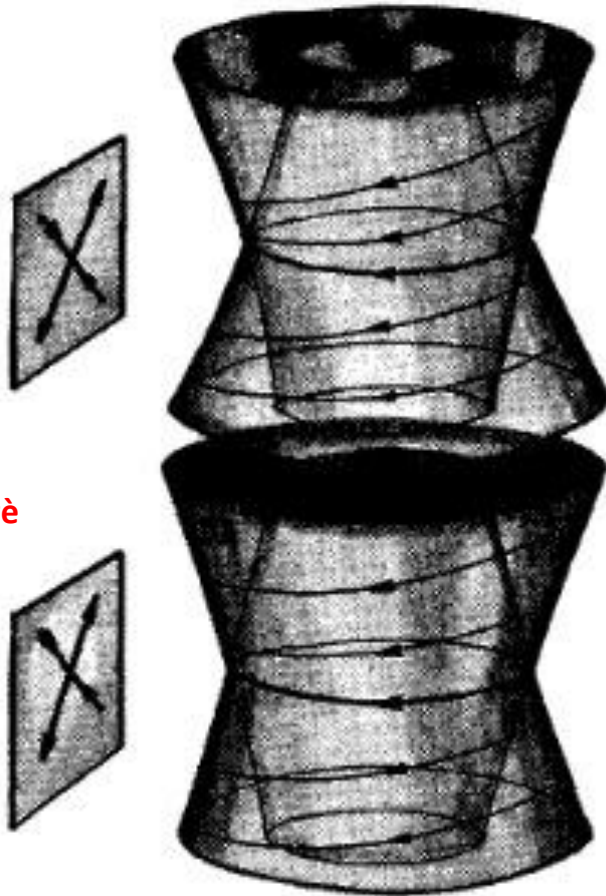
The same general type of behavior can result from a heteroclinic orbit, which connects one saddle point (or saddle cycle in the original state space) to another. A second heteroclinic orbit takes us from the second saddle point back to the first. When such a combined trajectory exists, we say we have a heteroclinic connection between the two saddle cycles. Figure 4.14 shows schematically a part of a heteroclinic connection in a Poincaré section of a three-dimensional state space. It is also possible to have heteroclinic orbits that link together sequentially three or more saddle cycles.

Fig. 4.14. The Poincaré section of a heteroclinic connection. Two saddle cycles intersect the plane at P and P' respectively. A heteroclinic orbit links together two saddle cycles forming a heteroclinic connection. For the sake of clarity only the part of the tangle involving the unstable manifold of P intersecting the stable manifold of P' is shown



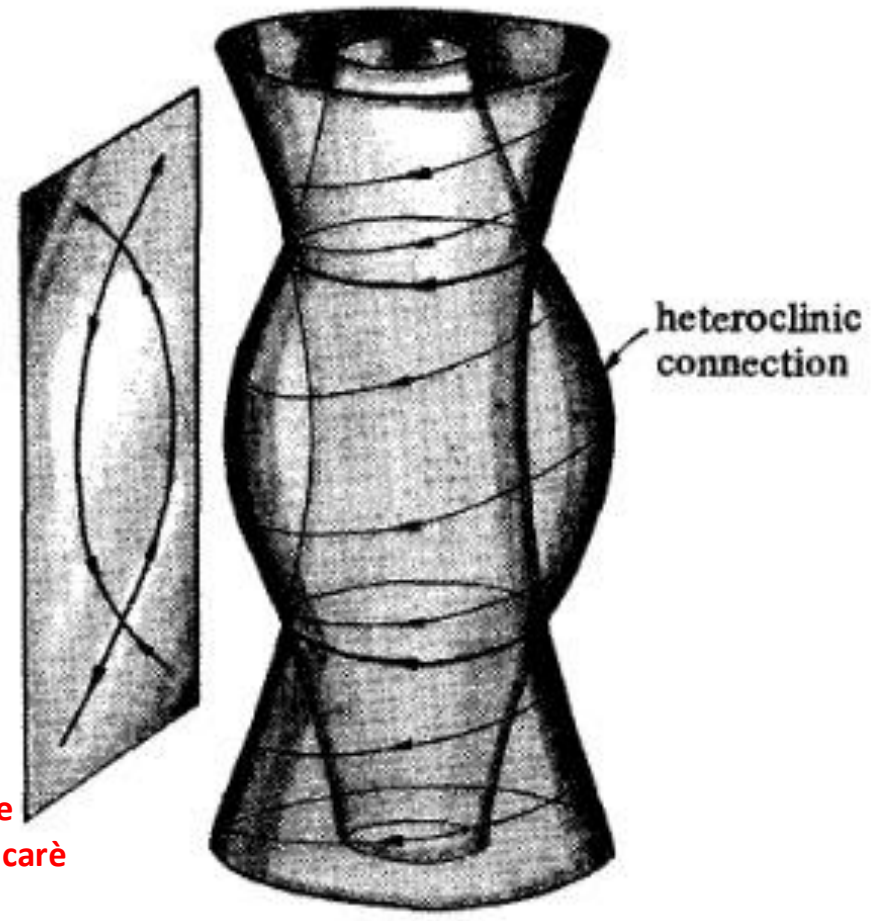
heteroclinic

connections in three-dimensional state spaces for which the in-sets and out-sets of a saddle cycle are two-dimensional surfaces. Also shown are partial pictures of the resulting Poincaré sections.



Sezioni
di Poincaré

(b)

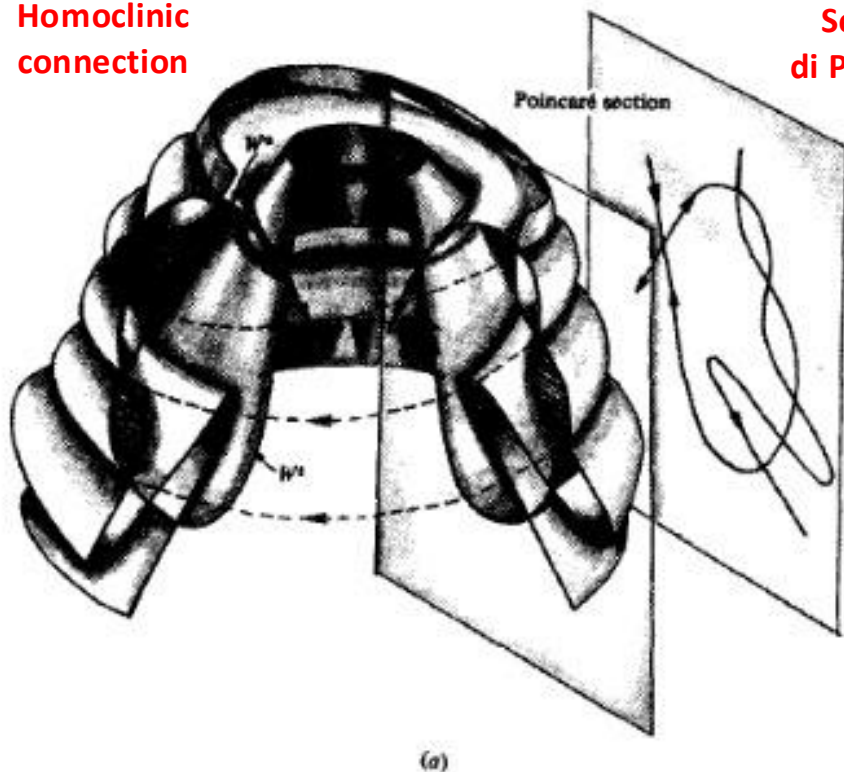


Sezione
di Poincaré

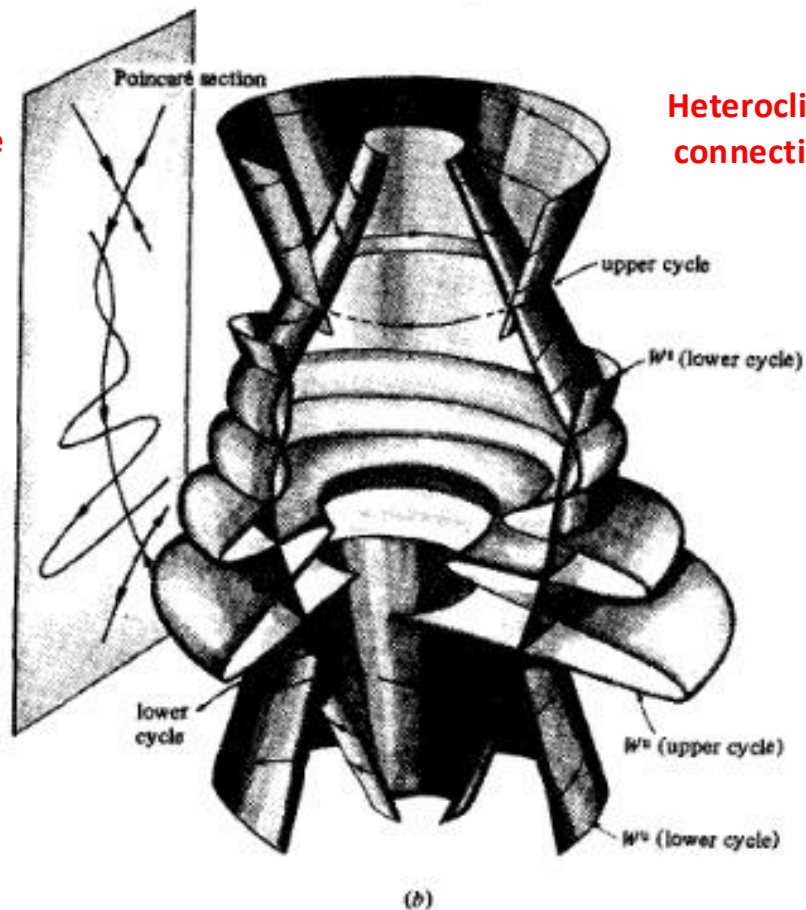
(c)

Figure 4.16 shows three-dimensional constructions of homoclinic and heteroclinic tangles resulting from the intersections of the in-sets and out-sets of saddle cycles. Partial diagrams of the corresponding Poincaré sections are also shown.

Homoclinic connection



Sezioni di Poincaré

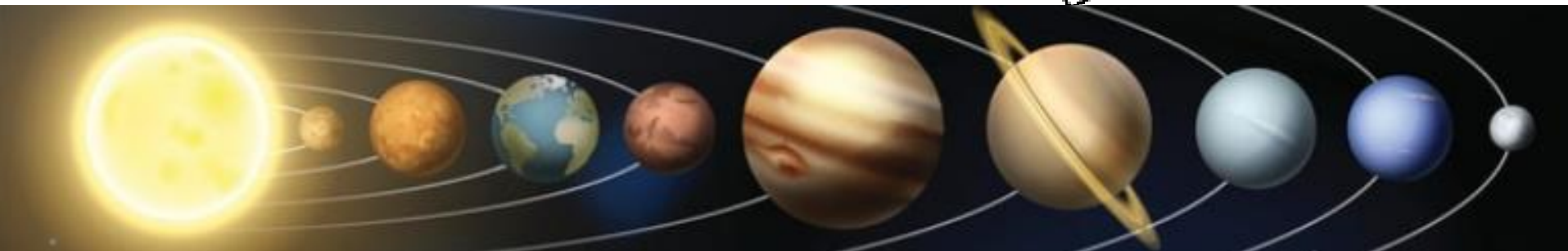
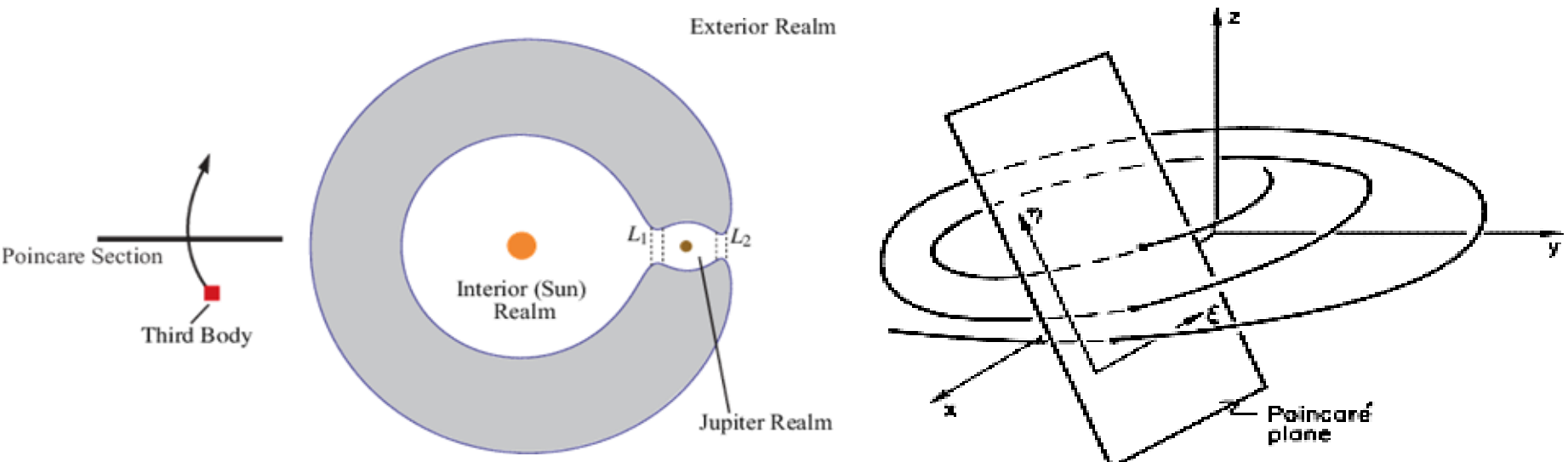


Heteroclinic connection



Così, nel terzo volume del suo *Méthodes nouvelles de la mécanique céleste* (1882-1889), lo stesso **Poincaré** descrisse la scoperta del «tangle» omoclinico compiuta mentre cercava di risolvere il problema dei tre corpi (un asteroide lanciato nel campo gravitazionale del Sole e di Giove) per mezzo della sua «sezione» (non avendo all'epoca i computer, ovviamente non poté mai visualizzarla...):

«Tentiamo di farci un'idea della figura formata da queste due curve e delle loro intersezioni, che sono in numero infinito e corrispondono ciascuna a una soluzione doppiamente asintotica; queste intersezioni formano una sorta di reticolo, di ordito, di rete dalle maglie infinitamente fitte; ciascuna delle due curve non deve mai intersecare se stessa, ma deve ripiegarsi su se stessa in maniera assai complicata per poter intersecare un'infinità di volte tutte le maglie della rete».





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Per approfondimenti: <https://www.vitapensata.eu/2022/01/04/tre-corpi-al-margine-del-caos/>

(Tre) corpi al margine del caos

Di: **Alessandro Pluchino**

4 Gennaio 2022

Come è noto, il Tre è spesso considerato il *numero perfetto* da diversi punti di vista: dal punto di vista *matematico* costituisce la sintesi dei pari (due) e del dispari (uno); dal punto di vista *esoterico* è il simbolo della Grande Triade (Cielo, Terra, Uomo); infine, dal punto di vista *religioso*, rappresenta la perfezione divina (si pensi alla Trinità del Cristianesimo o alla Trimurti induista). Pochi forse sanno, però, che allo stesso tempo il tre rappresenta anche la *soglia dell'imperfezione*, il numero magico che ha condotto la fisica moderna al confine tra ordine e disordine, in quella strana regione oggi conosciuta come "Margine del Caos", spalancando così le porte alla nuova Scienza della Complessità. E la scintilla da cui questa rivoluzione concettuale è partita riguardava un problema di corpi. Per la precisione, appunto, di tre corpi.



Tutto cominciò la notte tra il 31 agosto e il primo settembre del 1879 in una miniera di carbone di Magny, nella Borgogna francese. Alle 3.45 circa del mattino un'esplosione improvvisa scosse la miniera, ustionando e uccidendo gran parte della squadra di ventidue minatori che si trovavano al lavoro a quell'ora. Fu soltanto la perizia e l'acume scientifico di un giovane ingegnere incaricato delle indagini a permettere di risalire alla causa prima dell'esplosione: si era trattato di una lampada perforata accidentalmente che aveva lasciato uscire la fiamma da cui poi, a contatto con un'atmosfera ricca di metano come quella della miniera, aveva avuto inizio il processo che avrebbe portato alla conflagrazione. Quel giovane ingegnere, appena venticinquenne, si chiamava Jules-Henri Poincaré, colui che più avanti si sarebbe distinto come uno dei più grandi matematici e fisici di fine Ottocento (all'epoca si poteva essere ingegnere, matematico e fisico allo stesso tempo!) e che è considerato oggi uno dei padri della teoria dei sistemi dinamici e il precursore assoluto della moderna teoria del Caos. Sarà lui il principale protagonista della storia che stiamo per raccontarvi.



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